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Communications

Why Perturbations?

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Abstract**:** *Review on the book M. M. Konstantinov, P. H. Petkov. Perturbation Methods in Matrix Analysis and Control. NOVA Science Publishers, Inc., New York, 2020. ISBN 978-1-53617-470-0.*
https://novapublishers.com/shop/perturbation-methods-in-matrix-analysis-and-control/

The sensitivity of a given mathematical object (or of the corresponding computational problem) is among its most important properties. It shows how the solution of the problem varies under the perturbations of small changes in the data. This property is subject of the so called *Perturbation Theory* which is widely used in Science and Engineering. In the separate scientific disciplines various perturbation theories have been developed which differ in the problems solved and mathematical methods used, for instance in the Celestial Mechanics, Theory of Nonlinear Oscillations and Control Theory. All these theories are based on the idea to investigate a system whose behavior deviates slightly from the behavior of a simple ideal system for which the full solution of the problem under consideration is known. Perturbation Theory for Linear Operators, which is relevant to the given case, was created by the physicists Strutt and Lord Rayleigh [17] and Schrödinger [16] and the modern perturbation theory for linear operators is developed by Kato [7]. Although the book of Kato is written from functional analysis point of view, it contains useful material on the finite dimensional case as well.

There are at least three sound reasons to study the sensitivity of various problems relative to perturbations in the data from a given class.

- The perturbation analysis may give an independent and deep insight at the very nature of the problem, being therefore of independent theoretical interest. For example, the perturbation analysis may provide an estimate for the distance from an object of a given set, for instance the set of singular matrices or matrices with multiple eigenvalues [5].

- Perturbation bounds provide a realistic framework for most problems in mathematical modeling of objects and processes. Indeed, in practice there are inevitable measurement and other parametric and/or structural uncertainties. This means that we have to deal with a *family of models* rather than with a single model. In this case, the perturbation bounds give us *a tube* in the space of model's characteristics, to which the characteristics of the particular model actually belong. Having a model with given parameters and estimates for their values, the only thing that we can rigorously claim is that the model will behave within the tube predicted by the perturbation analysis.
- When a numerically stable algorithm is applied to solve a problem then the solution, computed in finite arithmetic, will be close to the solution of a near problem. Having tight perturbation bounds and a knowledge about the equivalent perturbation for the computed solution, it is possible to derive estimates of the conditioning of the problem and for the accuracy of the solution. Without such estimates a computational algorithm cannot be recognized as reliable from the viewpoint of modern computing standards.

The book under review, published in the series "Computational Mathematics and Analysis" is devoted to the perturbation theory applied in Matrix Analysis and Control Theory. The matrix perturbation theory is based on the matrix analysis as presented in the classical books of G a n t m a c h e r [6], L a n c a s t e r and T i s m e n e t s k y [14], and B e r n s t e i n [1]. The most comprehensive book up to the moment presenting the matrix perturbation theory is the book of S t e w a r t and S u n [20] and other useful sources in this field are the works of B h a t i a [2, 3]. The book of Konstantinov and Petkov is the first book which presents an original unified perturbation theory suitable for problems both in Matrix Analysis and Control Theory. The authors are well known specialists in this field who have published dozens of papers on the perturbation analysis of different problems in matrix analysis and control and co-authored the book [8]. They are also university teachers and have several successful students. The book presents a new and very efficient method for perturbation analysis called the Method of Splitting Operators and Lyapunov Majorants (MSOLM). This method was introduced in 1990 by the authors in [13, 15] and developed in several publications [9-12]. It is applicable to matrix objects and problems involving unitary (orthogonal in particular) transformations and matrices. It should be noted that such transformations are the "working horse" of the modern numerical methods for matrix computations. MSOLM is applied to the following important problems:

- QR decomposition of a general matrix,
- The Schur decomposition of a matrix,
- The generalized Schur decomposition,
- Condensed forms of Hamiltonian matrices,
- Unitary (orthogonal) canonical forms of linear time-invariant systems in standard and descriptor forms,
- Synthesis of state and output static feedback in linear time-invariant control systems,
- Spectral analysis of matrices and matrix pencils.

Since 1994 the MSOLM has been used and generalized by several scientists for obtaining new or improved asymptotic and nonlocal perturbation bounds for a number of matrix problems, see for instance [4, 21].

The book consists of 10 chapters, References and Index. Briefly, its contents is as follows. In Chapter 2 the authors give the notation and preliminaries used later on. The basic perturbation problems in linear algebra and control which can be successively solved by MSOLM are described in Chapter 3. In Section 7 of the same chapter they study problems with non-unique solution in view of their importance in matrix decompositions and in the problem of synthesis of static feedback (modal control in particular) in linear multivariable systems. Section 8 is devoted to the construction of asymptotic (linear) and nonlocal (nonlinear) perturbation bounds. In Chapter 4 authors give the general statement of the perturbation analysis problem and present the MSOLM. A particular problem (the QR decomposition of a matrix) is considered in detail in Section 6. The application of this method to the perturbation analysis of the Schur decomposition of a square matrix is demonstrated in Chapter 5. A considerable attention is paid to the perturbation theory of the condensed forms of Hamiltonian matrices which is done in Chapters 6, 7 and 8. The perturbation analysis of orthogonal canonical forms of linear control systems is presented in Chapter 9 and the general problem of synthesis of linear static feedback in linear control systems is considered in Chapter 10. The references contain 166 titles in the area of matrix analysis and perturbation theory.

The book is intended as a reference for specialists working in the fields of Matrix analysis and Control Theory. It may also be used in the development of specialized student courses in the above areas.

Consider now the main contributions presented in the book. The basic tool for perturbation analysis proposed by the authors in details in Chapter 4 is the Method of Splitting Operators and Lyapunov Majorants. An important step of this method is the construction of an operator equation, which is equivalent to the perturbed problem. It is based on the splitting of a certain linear matrix operator L and its argument X into strictly lower, diagonal and strictly upper parts L_1, L_2, L_3 and X_1, X_2, X_3, respectively. The crucial fact here is that for the problems under consideration $L_1(X)$ depends only on X_1 rather than on the whole matrix X. As an example, consider the perturbations in the factors Q and R of the QR factorization of a matrix where Q is unitary (orthogonal) and R is and upper triangular matrix. As it is well known, the QR decomposition is a basic tool in the rank analysis (including the numerical rank) of a matrix, as well as for reliable solution of linear algebraic equations, least squares problems and other important numerical problems. In this case the perturbations of the unitary factor Q are determined only by the zero strictly lower triangular part of the perturbed matrix R. Once we know the perturbations of Q, it is easy to find the perturbations of the upper triangular part of R. Fortunately, we have the same situation in all matrix and control problems listed above. The next step of MSOLM is the application of Lyapunov majorants. These majorants are used in the context of the analysis of operator equations arising in problems of nonlinear mechanics to establish existence and uniqueness conditions for the solutions of equations in functional spaces as a preliminary step in the use fixed point topological principles. In this book they are implemented to derive rigorous nonlinear perturbation bounds.

The cornerstone of the present book is the perturbation analysis of the Schur decomposition done in Chapter 5. As it is known, the Schur decomposition $A = UTU^{\mathrm{H}}$ where U is unitary and T is upper triangular matrix is widely used in matrix analysis and in the numerical solution of the eigenvalue problem for n-th order matrices. The first results in this area are obtained by Stewart [18, 19] but the corresponding analysis requires manipulation of matrices whose size is $n^2 \times n^2$. Using the MSOLM the authors derive for the first time norm-wise linear and nonlinear bounds for the perturbations in U and T which involve a matrix of order only $n(n-1) \times n(n-1)$. The bounds derived allow to introduce condition numbers for the corresponding quantities. The condition number reveals the sensitivity of the Schur form and allow to obtain accuracy estimates for the Schur decomposition computed in finite precision arithmetic. A fundamental result here are the estimates of the eigenvalue sensitivity which are found on the basis of perturbation bounds for T. The eigenvalue bounds and the corresponding eigenvalue condition numbers are determined without the usage of eigenvectors and may represent alternative to the well-known results in this important field. In my view, these new bounds can be put among the classical results of eigenvalue perturbation theory. This is a specific result for finding component-wise perturbation bounds for the Schur form, which as a whole is still an unsolved problem. It is my opinion that the results of this Chapter can be extended further to find new estimates for the sensitivity of invariant subspaces which will make possible to develop a full theory for the perturbation analysis of the Schur form of a matrix. It is interesting to note that the preliminary results on the norm-wise perturbation analysis of the Schur form published in [13] inspired the well-known researcher J.-G. Sun [21] to derive similar results for the generalized eigenvalue problem.

One of the important and unsolved until recently problems in the matrix perturbation theory is the sensitivity analysis of the condensed (and canonical, in particular) forms of Hamiltonian matrices in respect to unitary similarity transformations preserving their structure. In the given case the usage of the Schur form does not solve the problem since the similarity action of the full unitary group destroys the Hamiltonian structure. As it is known, in constructing and studying of structure-preserving methods for structured problems it is important to analyze the influence of perturbations that are also structured. This assumes investigation of the problem of sensitivity (including determination of condition numbers) and of accuracy of the corresponding numerical methods. In the case of spectral analysis of Hamiltonian matrices it is necessary to study the sensitivity of the Hamiltonian-Schur form relative to perturbations preserving the Hamiltonian structure. In the given case these are perturbations which are in turn Hamiltonian matrices. These problems are investigated in Chapters 6, 7 and 8 of the book. In Chapter 8 the authors derive asymptotic (linear) perturbation bounds and in Chapter 7 they present non-local (non-linear) bounds. The results obtained can be applied to various problems including the numerical analysis of matrix Riccati equations.

The next two Chapters 8 and 9 are devoted to perturbation problems arising in Control Theory and involving matrices. In Chapter 9 the authors present a perturbation analysis of the orthogonal canonical forms used in the analysis and design of linear control systems. After precise definition of the notion "canonical forms" they derive local and non-local estimates of the perturbations in single-input/single-output canonical forms. On the basis of these estimates they introduce for the first time condition numbers characterizing the sensitivities of the corresponding terms. The perturbation analysis, presented in Section 5 of this Chapter is much more difficult due to the non-uniqueness of the canonical form in this case. The results obtained for the multi-input case are based on a specific regularization technique using the perturbation in the system controllability matrix and the notion of a numerical structure of a multi-input system. In this case only perturbations preserving the numerical structure are considered. Numerical examples are given that illustrate the results of the analysis.

In the next Chapter 10 the authors present complete sensitivity analysis of the output and state feedback synthesis problems for linear multivariable systems, the pole assignment problem being studied in particular. Local linear and non-local nonlinear perturbation bounds are derived using the Schur form of the closed-loop state matrix. The local bounds are given not only in terms of condition numbers but using special homogeneous functions which give better results. The approach is based again on the technique of splitting operators which makes possible to get perturbation bounds for basic problems in matrix analysis and control theory.

In Summary, in this book the authors present new, original results in perturbation linear algebra and control, based on the Method of Splitting Operators and Lyapunov Majorant Functions. Combined with the Schauder or Banach fixed point principles, this method allows to obtain rigorous non-local perturbation bounds for a set of important objects in matrix analysis and control theory. Thus, the perturbation problems in these important fields are investigated in a uniform way, which presents a significant contribution to perturbation theory. As a direction of further work, I would recommend to extend the results obtained to the case of component-wise perturbation analysis in order to find perturbation bounds for the individual super-diagonal elements of the Schur form and for the angles between the perturbed and unperturbed invariant subspaces of the matrix. This will allow to develop full perturbation theory in this important form and go deeper into the properties of the corresponding problems.

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