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Chapter 19

THE RANDOM FLIGHT AND THE PERSISTENT RANDOM WALK

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Abstract

Almost all of this chapter is a review of the subject advertised in the title, although a few proofs or results are new. There are some references to applications at the end, but the point of view is the one of mathematical physics. The subject of persistent random walks is to broad to be given a fair review of a length appropriate to a chapter, so we have focused on the essentials, which is more or less what is understood by "random flight". Although both names refer to the same topic, as explained in this chapter, it has been studied by many physicists under the name "persistent random walk", and they naturally tend to find applications and extend their field of study. On the other hand mathematicians have studied the topic under the name "random flight", and they have stayed closer to the original problem.

The subject of study of this chapter is clearly stated in its first paragraph. Then the first few sections build up progressively to it, while discussing physical models to have a concrete as possible picture of the subject. The main part of the chapter shows the methods that have been applied to solve the problem and the results obtained. The most characteristic feature of this chapter is the central position occupied by the Born expansion, which is an expansion for the solution to our problem indexed by the number of collisions. All the results that have been obtained by other methods can be obtained using the Born expansion, which also gives a physical picture. The last three sections are extensions of the problem stated in the first paragraph of the chapter. The first of these sections, "Anisotropic scattering", is useful to firmly bridge what is understood by "random flight" and by "persistent random walk". The choice of the second of these sections, "Projections onto lower dimensional spaces" is due both to the taste of the author and to the very recent developments. The last section mentions all else that should have been included in a review on the persistent random walk.

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1. Introduction

Most of this chapter is devoted to the following process:

A particle starts at the origin of coordinates, moves with a constant speed v *in a straight line along a random direction. Then it changes direction at some random (Poisson) instant and takes another direction. The change of direction is instantaneous and takes place at a point. The speed* v *between the changes of direction is the same throughout the process. The direction which the particle takes at each change is isotropically distributed.*

This process has been called "random flight" (chapter 2 of [1]). It is equivalent to the "persistent random walk" [2]. This equivalence will be explained qualitatively in model 2 of section 4 and quantitatively in section 9.

To have a physical model of this process in mind we may, tentatively, assume that the changes of direction undergone by the particle are due to the interaction with randomly (Poisson) laid scatterers of infinite mass, so that they remain still during the scattering and do not change the energy (speed) of the scattered particle. It is good to have this physical model in mind, as long as one is aware that the random flight is a simplified picture of it. The reasons why this is a simplified picture as well as other models will be discussed in section 4.

The goal is to find the probability density $\rho(t, \vec{x})$ of a particle to be at time t at position $\vec{x}$. It is equivalent, but easier, to picture many particles which leave the origin of coordinates isotropically with speed v and become scattered without interacting with each other. Then $\rho(t, \vec{x})$ becomes the density of particles at time t at position $\vec{x}$. This ensemble of particles is a spherically expanding front first, before any scattering has taken place. Then some particles become scattered and lag behind the expanding front because they have taken a direction which is not the outgoing direction any more. In this way, little by little, density is transferred inwardly from the expanding front. When the mean number of scatterings undergone by each particle has become large, the density inside the front is largest at the origin and decays towards the expanding front. The graph of this density has some bell looking shape, which tends to a Gaussian, as we shall see. In addition, some particles will still remain unscattered and constitute the expanding spherical front. The case of non isotropic initial conditions (i. e., the initial cloud does not have spherical symmetry) can be put, as we shall see later, in terms of the case of isotropic initial conditions.

In the isotropic scattering the particle scatters with equal probability along any direction. When the scattering is anisotropic, the probability depends on the angle with respect to the incoming direction. Much is known analytically about the isotropic scattering and little about the anisotropic scattering. As implied by the first sentence of this chapter, we review the anisotropic scattering only very superficially, but the anisotropic scattering in 1 dimension is explained in detail to show the quantitative connection between the definitions of random flight and persistent random walk.

In sections 2-4 we start with the simplest model of random walk and progressively generalize it to the random flight. In sections 5-8 we review the methods and results on the random flight. Section 9 is about the anisotropic scattering, in section 10 we show that the solution of the random flight problem in a given dimension solves other random walk problems in lower dimensions and in section 11 we mention unreviewed areas in the persistent random walk.

Almost all of the chapter is a review, but the following three things are new. The proof of the weak convergence of $\rho(t, \vec{x})$ to a Gaussian in d dimensions when $t \to \infty$ in subsection 6.2. The proof that the telegrapher's equation solves the random flight in 1 dimension in subsection 8.1. The first paragraph of section 10, which shows that projection of a random flight onto an Euclidean subspace is also a (generalized) random flight. Section 4, the solution of the anisotropic case in 1 dimension, the discussion of the "bullet initial condition" in section 5, expression (47) for the solution of the 1 dimensional case with isotropic initial condition and the solution of the 1-dimensional case for the "bullet initial condition" at the end of subsection 8.1 are perhaps new.

2. The Simple Random Walk

In the so called [3] "simple random walk" time and space are discrete and space is 1-dimensional. Let τ be the time step and $v\tau$ the space step, where v is the speed of the particle. In the simple random walk if the particle is at time t at position x, then it will be at time $t + \tau$ at position $x \pm v\tau$ with probability 1/2. t is a multiple of τ. The main questions about this model had been posed and answered in the form of questions about gambling [4] before atomic theory allowed random walks to be a subject of Physics. We are going to give the solution of this model using notation which is cumbersome, but which will make easier the transition to the more complicated models which will follow. We define

$$s(\tau, x) = \frac{1}{2}\big(\delta(x - v\tau) + \delta(x + v\tau)\big). \tag{1}$$

Let the particle be at site x' and time t with probability $\rho(t, x')$. This probability splits in two halves and migrates to $x' \pm 1$ at time $t + \tau$. This is the content of the following equation, where the Dirac deltas in the preceding equation allow the use of the integral notation.

$$\rho(t+\tau, x) = \sum_{x' \in Z} \rho(t, x')\, \frac{1}{2}\, \left(\delta_{x-x'-1,0} + \delta_{x-x'+1,0}\right) = \int dx'\, \rho(t, x') s(\tau, x - x') =$$

$$\rho(t, \_) \otimes_s s(\tau, \_)(x) \tag{2}$$

where $\rho(t, \_) \otimes_s s(\tau, \_)(x)$ is the space convolution of the functions $\rho(t, \_)$ and $s(\tau, \_)$ of x. It follows from the last expression by induction that

$$\rho(t, x) = \otimes_s^{t/\tau} s(\tau, \_)(x). \tag{3}$$

The notation $\otimes_s^{t/\tau} s(\tau, \_)$ means $t/\tau - 1$ space convolutions, so that the function ($s(\tau, \_)$ in our case) to be convoluted appears t/τ times. This is consistent with the usual exponential notation for numbers, where to raise a number to the n-th power, $n - 1$ multiplications have to be done.

If the particle starts at the origin then the preceding formula yields

$$\rho(t, x) = \frac{1}{2^{t/\tau}} \binom{t/\tau}{(t/\tau + x/(v\tau))/2}. \tag{4}$$

This distribution is the binomial distribution with support on the even or odd sites of vZ if parity of t/τ is even or odd, respectively. If the parities of t/τ and $x/(v\tau)$ are different, then the probability is 0. There are two differences between the binomial number in this density and $\binom{t/\tau}{x/(v\tau)}$. One is that the density's center is at $x = 0$, whereas the binomial's center is at $x = (t/\tau)/2$. The other is that the density is spread by a factor of two with respect to the binomial, because in the former, at any given time, only the odd or even sites support the density.

In a slightly more complex model

$$s(\tau, x) = P\delta(x - v\tau) + Q\delta(x + v\tau), \quad \text{where} \quad P, Q \geq 0, \;\; P + Q = 1. \tag{5}$$

In this case one says that there is a drift or that the random walk is biased. If the particle starts at the origin then formula (4) yields

$$\rho(t, x) = \binom{t/\tau}{(t/\tau + x/(v\tau))/2} P^{(t/\tau + x/(v\tau))/2} Q^{(t/\tau - x/(v\tau))/2}. \tag{6}$$

If we define $R \equiv P - Q$, then the first two moments of the preceding binomial are

$$\mu(t) = (P - Q)vt = Rvt \tag{7}$$

and

$$\sigma^2(t) = 4PQv^2\tau t = (1 - R^2)v^2\tau t \tag{8}$$

Any macroscopic measurement of the binomial distribution is a function of its first and second moment. Thus the experimental information is Rv and $(1 - R^2)v^2\tau$ and the microscopic parameters are: R, v and τ. If one of the microscopic parameters is known the other two can be found.
The quantity $\frac{(1-R^2)v^2\tau}{2}$ is called the diffusion coefficient, D. When there is no drift, $R = 0$ and the experimental information is

$$2D = v^2\tau = v\ell, \tag{9}$$

where ℓ is the distance between the scatterers.

In an d-dimensional cubic lattice the function s becomes

$$s(t, \vec{x}) = \frac{1}{2d} \sum_{i=1}^{d} \delta(\vec{x} \pm v\tau\vec{e}_i), \tag{10}$$

where $\vec{e}_i$ is the i-th basis vector. Formulae (2) and (3) hold and the solution is given by more complicated combinatorial expressions.

It is sometimes presented as a weakness of this model and its extensions to square and cubic lattices that it does not have a continuum limit, because if $\tau \to 0$, then $v \to \infty$ in order to keep the diffusion coefficient, which is the observable quantity, constant. τ, however, has the interatomic distance divided by v as a lower bound, so this model is not inconsistent.

3. The Pearson Random Walk

In the Pearson random walk scattering still takes place at multiples of a given step of time, τ, but space is continuous. Then, if $\rho(t, \vec{x})$ is the probability density at $(t, \vec{x}),\ \vec{x} \in R^d$, the probability density at time $t + \tau$ is

$$\rho(t + \tau, \vec{x}) = \int d\vec{x}'\ \rho(t, \vec{x}')s(\tau, \vec{x} - \vec{x}') \tag{11}$$

where

$$s(t, \vec{x}) = \frac{\Gamma(d/2)\delta(r - vt)}{2\pi^{d/2}(vt)^{d-1}} \quad \text{and} \quad r = |\vec{x}|. \tag{12}$$

$\frac{2\pi^{d/2}}{\Gamma(d/2)}(vt)^{d-1}$ is the surface of an d-dimensional sphere. It follows again by induction that

$$\rho(t, \vec{x}) = \otimes_s^{t/\tau} s(\tau, \_)(\vec{x}), \tag{13}$$

where $\otimes_s^t s(\tau, \_)$ means t/τ times the space convolution of the function of space which is the function s restricted to time $t = \tau$. Solution of the Pearson random walk amounts, then, to convoluting spherical surfaces of the same radius. A simple analytical expression for this is known only in 1 dimension, as shown in the previous section, where the Pearson random walk is the simple random walk. For a review of the Pearson random walk see [5].

4. Models of Persistent Random Walk

The next natural generalization of the random walk is to let time be continuous and to let scattering take place at any time with constant probability, instead of only at regular intervals of time. This is the basic process defined at the beginning of the chapter. Before solving the basic process we discuss the physical model for it proposed in the second paragraph of this chapter, henceforth to be called model 1, as well as three other models.

Physical scatterers are not pointlike, but occupy a finite volume. Therefore scatterers cannot be Poisson distributed at scales shorter than twice the radius of that volume. A more subtle consequence of the finite size of the scatterers is that the probability that the particle hits some scatterer more than once is non-zero. When that happens, if the particle takes a direction close enough to some previous time, then it will hit the next scatterer not after an exponentially distributed interval of time, but after the same interval as the last time. Randomness in space, then, does not imply randomness in time any more. This effect makes the basic process not fit model 1 in 1 dimension, regardless of the size of the scatterer. In dimension greater than 1 the basic process is an approximation to model 1 which is better in dimension 3 than 2 and better when the density and the size of the scatterers diminish.

Using the language of statistical mechanics one may say that the random flight has annealed disorder, while the model 1 has quenched disorder. This problem is avoided in the three models that follow.

Model 2. Here the density of scatterers is large and the mean free time between collisions, τ', is small. The angular cross section has an isotropic component and a large forward scattering component. To an outside observer who cannot see the scatterers it is as if the particle only met scatterers once in a while (with a mean free time between collisions $\tau >> \tau'$) and scattered isotropically. This equivalence is quantitatively presented in the section on anisotropic scattering.

This model is the origin of the name "persistent random walk" [6, 7]. Here "persistent" means "with forward scattering", as opposed to the simple or the Pearson random walk.

Model 3. Here the scatterers are laid in some periodic array, so that there is no difference in arriving to a visited or unvisited scatterer. The distribution of distance, however, is not quite an exponential.

Model 4. Here the scatterers are in motion, so that the particle cannot find itself again in some previous situation.

5. Initial Conditions

The persistent random walk will be solved for two different initial conditions: the isotropic initial condition and the "bullet" initial condition (we called it "stream initial condition" in [8], but "bullet initial condition" is more apropriate). If no scatterers were present, the solutions for each initial condition would be, respectively,

$$s(t,\vec{x}) = \frac{\Gamma(d/2)\delta(r-vt)}{2\pi^{d/2}(vt)^{d-1}} \quad \text{and} \quad r = |\vec{x}|. \tag{12}$$

and

$$b(t,\vec{x}) = \delta(x_1')\cdots\delta(x_{d-1}')\,\delta(x_\Omega - vt). \tag{14}$$

In the second case Ω is some d-dimensional direction and the primed directions are orthogonal to the direction Ω. This defines both initial conditions.

The persistent random walk will be solved using three different methods, which are presented in the next three sections. For the first two methods the initial conditions are used in the form above. For the last method, which consist in solving a differential equation, we need the value and the time derivative at $t = 0$ of the above functions.

When the initial condition is isotropic,

$$\rho(0,\vec{x}) = s(0,\vec{x}) = \delta(\vec{x}) \tag{15}$$

and

$$\frac{\partial\rho}{\partial t}(0,\vec{x}) = \frac{\partial s}{\partial t}(0,\vec{x}) = 0. \tag{16}$$

To understand the last equality, notice (see eq. (12)) that $\frac{\partial s}{\partial t}(0,\vec{x})$ is zero except possibly at $\vec{0}$. But for $t < 0$, s is an ingoing spherical wave and $s(t,\vec{0}) = 0$. For $t > 0$, s is an outgoing spherical wave and $s(t,\vec{0}) = 0$. At $t = 0$, $s(t,\vec{0})$ is infinite. It follows that $s(t,\vec{0})$ has a maximum at $t = 0$ and its derivative there is 0. This argument can be transformed into a rigorous proof by using the $\delta(x) = \lim_{\sigma\to 0} \frac{\exp(-x^2/(2\sigma^2))}{\sqrt{2\pi\sigma^2}}$ representation of the Dirac delta function and using the symmetry of the function $s(t,\vec{x})$ with respect to time inversion at

$t = 0$. Conditions (15) and (16) have been proven using different methods ([9], section 4, remark 5).
When the unscattered probability density is the bullet initial condition in the Ω direction,

$$\rho(0, \vec{x}) = b(0, \vec{x}) = \delta(\vec{x}) \tag{17}$$

and

$$\frac{\partial \rho}{\partial t}(0, \vec{x}) = \frac{\partial b}{\partial t}(0, \vec{x}) = -v\delta(x'_1) \cdots \delta(x'_{d-1})\delta'(x_\Omega). \tag{18}$$

It is obvious from Physics that, if the positions of the scatterers are given, then the initial position $\vec{x}_0$ and velocity $v\vec{e}_\Omega$ (where Ω is some d-dimensional direction) of the particle will determine $\rho(t, \vec{x})$. From a mathematical point of view any unscattered probability density is a linear combination of bullet initial conditions. The linearity of the three methods which follow show that solving the bullet initial condition case for arbitrary Ω solves the problem for any initial condition.

Though not obvious, it turns out that, conversely, the solution to the bullet initial condition case can be put in terms of the solution to the isotropic case, as we shall see in the next section.

6. The Born Expansion

The discussion of the Pearson random walk shows that if the particle scatters not at multiples of a step of time τ, but at time steps of duration $\tau_1, \cdots, \tau_c$, then

$$\rho(t, \vec{x}) = s(\tau_1, \_) \otimes_s \cdots \otimes_s s(\tau_c, \_)(\vec{x}) \tag{19}$$

if the initial condition is isotropic and

$$\rho(t, \vec{x}) = b(\tau_1, \_) \otimes_s s(\tau_2, \_) \otimes_s \cdots \otimes_s s(\tau_c, \_)(\vec{x}). \tag{20}$$

for the bullet initial condition. $s(t, \vec{x})$ is given by (12) and $b(t, \vec{x})$ is given by (14). Unless otherwise stated, we restrict the exposition to the isotropic initial condition case. If it is known that the particle has suffered collisions at instants $t_1, \cdots, t_c$ prior to time t, a mere change of notation shows that

$$\rho(t, \vec{x}) = s(t_1, \_) \otimes_s s(t_2 - t_1, \_) \cdots \otimes_s s(t - t_c, \_)(\vec{x}). \tag{21}$$

To find the probability density ρ_c when the particle has suffered c collisions at random instants prior to time t, we have to average over the probability densities corresponding to all possible c-tuples $t_1, \cdots, t_c$. This average is obviously proportional to

$$\int_0^{t_2} dt_1 \cdots \int_0^{t_c} dt_{c-1} \int_0^t dt_c \big(s(t_1, \_) \otimes_s \cdots s(t_c - t_{c-1}, \_) \otimes_s s(t - t_c, \_)\big)(\vec{x}). \tag{22}$$

The upper limits ensure that the inequalities $0 \le t_1 \le t_2 \le \cdots \le t_c \le t$ hold. But the subset of R^c that satisfies these inequalities is just the support of the integrand, because

$s(t,\vec{x}) \propto \delta(r-vt)$, which is zero when $t<0$. Therefore if the upper limits are changed to infinity, the value of the integral will not change.
To find the norm of the above expression, remember that the space convolution of two normalized functions is also a normalized function. Then its norm is

$$\int_0^{t_2} dt_1 \cdots \int_0^{t_c} dt_{c-1} \int_0^t dt_c \times 1 = \frac{t^c}{c!}. \tag{23}$$

and

$$\rho_c(t,\vec{x}) = \frac{c!}{t^c} \int_0^\infty dt_1 \cdots \int_0^\infty dt_{c-1} \int_0^\infty dt_c \big(s(t_1,\_) \otimes_s \cdots s(t_c - t_{c-1},\_) \otimes_s s(t-t_c,\_)\big)(\vec{x}) =$$

$$\frac{c!}{t^c} \otimes^{c+1} s(t,\vec{x}), \tag{24}$$

where $\otimes$ denotes space-time convolution.

In a Poisson process in which the mean time between events is τ, the probability of c events happening in an interval t is $e^{-t/\tau}\frac{t^c}{\tau^c c!}$. Therefore, the probability density for the isotropic initial condition case (henceforth denoted by ρ_s) is

$$\rho_s(t,\vec{x}) = \sum_{c=0}^{\infty} e^{-\frac{t}{\tau}} \frac{t^c}{\tau^c c!}\, \rho_c(t,\vec{x}) = e^{-\frac{t}{\tau}} \left(\sum_{c=0}^{\infty} \left(\frac{s\,\otimes}{\tau} \right)^c s \right)(t,\vec{x}). \tag{25}$$

This expansion indexed by the number of collisions has been called the Born expansion [10] because it is very similar to the expansion indexed by the number of collisions used in Optics or Quantum Mechanics (see, e. g., [11]). It has been used by some authors since at least 1987 [12].

The zeroth term of this expansion is

$$e^{-\frac{t}{\tau}} s(t,\vec{x}) = e^{-\frac{t}{\tau}} \frac{\Gamma(d/2)}{2\pi^{d/2}(vt)^{d-1}}\, \delta(r-vt), \tag{26}$$

which, in the ensemble picture of the basic process described in the fourth paragraph of this chapter, is the spherically expanding front built by the particles which still remain unscattered.

A review of the derivation of this formula shows that in the bullet initial condition case, the probability density (henceforth denoted by ρ_b) is

$$\rho_b(t,\vec{x}) = e^{-\frac{t}{\tau}} \left(\sum_{c=0}^{\infty} \left(\frac{s\,\otimes}{\tau} \right)^c b \right)(t,\vec{x}). \tag{27}$$

The zeroth term of this expansion is

$$e^{-\frac{t}{\tau}} b(t,\vec{x}) = e^{-\frac{t}{\tau}} \delta(x_1') \cdots \delta(x_{d-1}')\, \delta(x_\Omega - vt), \tag{28}$$

which represents the particles which still remain unscattered.

Likewise, for an arbitrary initial condition u,

$$\rho_u(t,\vec{x}) = e^{-\frac{t}{\tau}} \left(\sum_{c=0}^{\infty} \left(\frac{s\otimes}{\tau} \right)^c u \right)(t,\vec{x}). \tag{29}$$

where the zeroth term has the interpretation that we have seen in the two previous cases. We can now see how the solution of the bullet initial condition case and, therefore, the solution for any initial condition u, can be put in terms of the solution of the isotropic initial condition case. From the expressions for ρ_s and ρ_u it follows that

$$\rho_u(t,\vec{x}) = e^{-\frac{t}{\tau}} u(t,\vec{x}) + e^{-\frac{t}{\tau}} \left(\frac{\rho_s}{\tau}\, e^{\frac{\cdot}{\tau}} \otimes u \right)(t,\vec{x}), \tag{30}$$

where $e^{\frac{\cdot}{\tau}}$ denotes the function of time which takes the value $e^{\frac{t}{\tau}}$ at t. In the section "The integral equation" a more explicit expression for the case of bullet initial condition will be given.

Just as, from a mathematical point of view, the Pearson random walk amounted to the computation of the convolution of spherical surfaces, the random flight amounts to the computation of the convolution of conical surfaces. This is so because an expanding spherical surface in space is a cone in space-time, with the time axis as its axis of symmetry. This is perhaps the reason that, as noted in [13] (Remark 2.4), more analytical results have been obtained for the random flight than for the Pearson random walk.

Before attempting to compute the terms of the Born expansion and adding them, in the next two subsections we find the even moments of ρ (which determine ρ) in any dimension and the asymptotic limit of ρ in any dimension.

6.1. The Even Moments

The m-th radial moment of $s(t,\_)$ is $(vt)^m$. This simple result allows the analytic determination of the even radial moments of $\rho_s(t,\_)$. We omit the calculations, which are lengthy ([10], section 2). The result is:

$$\langle r^{2m} \rangle = (2m)!\, e^{-\frac{t}{\tau}}\, (vt)^{2m} \sum_{c=0}^{\infty} \left[\frac{\Gamma(d/2)}{\Gamma(1/2)} \right]^c \frac{(t/\tau)^c}{(2m+c)!}\, \sigma(m,d,c), \tag{31}$$

where

$$\sigma(m,d,c) = \sum_{\substack{i_1,\cdots,i_{c+1}\in N \\ i_1+\cdots+i_{c+1}=m}} \frac{\Gamma(m+d/2)}{\Gamma(i_1+d/2)\cdots\Gamma(i_{c+1}+d/2)} \frac{\Gamma(i_1+1/2)\cdots\Gamma(i_{c+1}+1/2)}{\Gamma(m+1/2)}. \tag{32}$$

and the term of index c is the one that corresponds to c collisions.

In one dimension

$$\sigma(m,1,c) = \sum_{\substack{i_1,\cdots,i_{c+1}\in N \\ i_1+\cdots+i_{c+1}=m}} 1 = \binom{m+c}{c} \tag{33}$$

and formula (31) has the simple form

$$\langle r^{2m}\rangle = \frac{(2m)!}{m!}\, e^{-\frac{t}{\tau}}\, (vt)^{2m} \sum_{c=0}^{\infty} \frac{(t/\tau)^c}{c!}\, \frac{(m+c)!}{(2m+c)!} = e^{-\frac{t}{\tau}}\, (vt)^{2m}{}_1F_1(1+m, 1+2m, \lambda t), \tag{34}$$

where ${}_1F_1$ is a hypergeometric function.

The support of the probability density $\rho_s(t,)$ is the sphere of radius vt. Since it is of bounded support, it is determined by its moments. Since $\rho_s(t,)$ can be thought of as a function of r^2, it is determined by its even moments alone ([14], theorem 3).

6.2. Asymptotics

The second moment is ([10], section 3)

$$\langle r^2\rangle = 2(v\tau)^2 \left(e^{-\frac{t}{\tau}} - 1 + \frac{t}{\tau}\right), \tag{35}$$

a result which holds for any dimension. This expression has the ballistic limit

$$\lim_{t\to 0}\langle r^2\rangle = (vt)^2. \tag{36}$$

The interpretation of this result is that as $t \to 0$ only the zeroth term of the Born expansion is present. The other limit is

$$\lim_{t\to\infty}\langle r^2\rangle = 2v^2\tau t = 2v\bar{\ell}t. \tag{37}$$

This limit also holds in one dimension, so it can be compared with formulae (8) and (9). The comparison shows that the two models are different not only at short times, but also asymptotically, and that scattering periodically with a period τ or randomly with a mean period τ, will not yield the same coefficient of diffusion. *In one dimension particles diffuse by a factor of $\sqrt{2}$ faster when the scattering takes place at random times than when it takes place periodically.*

The fourth moment is

$$\langle r^4\rangle = 4!(v\tau)^4 \left[-3 + \frac{t}{\tau} + 3e^{-\frac{t}{\tau}} + 2\frac{t}{\tau}e^{-\frac{t}{\tau}} + \frac{1}{2}\left(\frac{t}{\tau}\right)^2 e^{-\frac{t}{\tau}}\right] +$$
$$4!\,\frac{2+d}{6d}\,(v\tau)^4 \left[12 - 6\frac{t}{\tau} + \left(\frac{t}{\tau}\right)^2 - 12\, e^{-\frac{t}{\tau}} - 6\frac{t}{\tau}\, e^{-\frac{t}{\tau}} - \left(\frac{t}{\tau}\right)^2 e^{-\frac{t}{\tau}}\right] \tag{38}$$

As expected,

$$\lim_{t\to 0}\langle r^4\rangle = (vt)^4. \tag{39}$$

The other limit is

$$\lim_{t\to\infty}\langle r^4\rangle = \frac{2+d}{d}(2v^2\tau t)^2 = \frac{4(2+d)}{d}v^2(\bar{\ell})^2t^2. \tag{40}$$

This is, in d dimensions, the fourth radial moment of a normalized Gaussian whose second radial moment is the one obtained earlier, $\lim_{t\to\infty}\langle r^2\rangle = 2v^2\tau t = 2v\bar{\ell}t$. Does this happen for all the even moments? If it does, does ρ tend to a Gaussian as $t \to \infty$? The answer to both questions is yes. This was proven for the two-dimensional case using techniques of probability theory [15] and was shown for the one- [16], two-dimensional [17] case by applying Tauberian theorems to the Fourier-Laplace transform of ρ_s. Boguñá *et alii* [18] have shown that the limit of the moments of ρ_s are Gaussian for arbitrary dimension, again using Tauberian theorems to find the asymptotic form of the moments. Kolesnik has given a proof of the last equality of this section using the characteristic function [9]. We give here a proof that requires to see a pattern in the expression of the even moments of ρ, but is conceptually simple.

To see this pattern the sixth moment has been computed in detail in Appendix I (the fourth moment is perhaps too trivial to clearly see the pattern that allows to find the $\lim_{t\to\infty}$ limit). The limits are

$$\lim_{t\to 0}\langle r^6\rangle = (vt)^6. \tag{41}$$

and

$$\lim_{t\to\infty}\langle r^6\rangle = \frac{(4+d)(2+d)}{d^2}(2v^2\tau t)^3. \tag{42}$$

which is, in d dimensions, the sixth radial moment of a normalized Gaussian whose second radial moment is the one obtained earlier. Reviewing the computation of the sixth moment it is easy to see that

$$\lim_{t\to\infty}\langle r^{2m}\rangle = \frac{(2m)!}{m!}\frac{(2m+d-2)(2m+d-4)\cdots d}{(2m-1)!!\,d^m}\,v^{2m}\tau^m t^m =$$

$$\frac{(2m+d-2)(2m+d-4)\cdots d}{d^m}\,(2v^2\tau t)^m. \tag{43}$$

On the other hand, the $2m$-th moment of the d-dimensional Gaussian $\frac{\exp -(1/2)(r/\sigma)^2}{(2\pi\sigma^2)^{(d/2)}}$ is

$$\frac{2\pi^{d/2}}{\Gamma(d/2)(2\pi\sigma^2)^{d/2}}\int_0^\infty dr\, r^{d-1+2m}e^{-\frac{1}{2}\frac{r^2}{\sigma^2}} =$$

$$(2m+d-2)(2m+d-4)\cdots d\,\sigma^{2m} = \frac{(2m+d-2)(2m+d-4)\cdots d}{d^m}\,(d\,\sigma^2)^m, \tag{44}$$

where $d\ \sigma^2$ is the second moment of the d-dimensional isotropic Gaussian. The d-dimensional isotropic Gaussian is also determined by its even moments ([14], theorem 2). Therefore, theorem 6.16 of reference [19] (which states, roughly speaking, that convergence of the moments implies convergence in distribution) applies and it follows that the probability density $\rho_s(t,\_)$ converges in distribution to the d-dimensional Gaussian as $t\to\infty$, i. e.,

$$\lim_{t\to\infty} \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^r dr'\ r'^{d-1} \rho_s(t,r') = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^r dr'\ r'^{d-1} \left(\frac{d}{4\pi v^2 \tau t}\right)^{d/2} \exp -\frac{d}{2}\frac{r'^2}{2v^2\tau t}. \tag{45}$$

6.3. One Dimension

If the c-th space-time convolution of s is known and the Born expansion can be summed, then ρ can be found. The content of formula (16) of [20] is essentially the c-th space-time convolution of $s(t,x) = \frac{1}{2}\big(\delta(x-vt)+\delta(x+vt)\big)$ (note that that expression is actually independent of η, our $1/\tau$). We do the change of notation $m \to c,\ l \to vt$, multiply by $\frac{t^c}{c!}$ (because expression (16) of [20] is normalized) and take the absolute value of x (because expression (16) of [20] yields the correct result only for positive x). Then

$$s^{\otimes(c+1)}(t,x) = \frac{1}{2}\frac{1}{c!(2v)^c} \Bigg(\delta(x+vt)(vt)^c + \delta(x-vt)(vt)^c + \sum_{k=1}^{c} \frac{(c+k)!}{k!(c-k)!(k-1)!2^k} |x|^{c-k}(vt-|x|)^{k-1} H(vt-|x|)\Bigg), \tag{46}$$

where H is the Heaviside function. Substitution in expression (25) yields

$$\rho_s(t,\vec{x}) = \sum_{c=0}^{\infty} e^{-\frac{t}{\tau}} \frac{\otimes^{c+1} s(t,\vec{x})}{\tau^c} =$$

$$e^{-\frac{t}{\tau}} \sum_{c=0}^{\infty} \left(\frac{t^c}{c!(2\tau)^c}\, s(t,x) + \frac{1}{c!(2v\tau)^c} \sum_{k=1}^{c} \frac{(c+k)!}{k!(c-k)!(k-1)!2^k} |x|^{c-k}(vt-|x|)^{k-1} H(vt-|x|) \right) =$$

$$e^{-\frac{t}{2\tau}}\, s(t,x) + \frac{e^{-\frac{t}{\tau}}}{2(vt-|x|)} \sum_{c=1}^{\infty} \frac{1}{c!} \frac{|x|^c}{(2v\tau)^c} \sum_{k=1}^{c} \frac{(c+k)!}{k!(c-k)!(k-1)!} \left(\frac{vt-|x|}{2|x|}\right)^k H(vt-|x|). \tag{47}$$

6.4. Two and Three Dimensions

ρ_s for two dimensions was obtained by Stadje [12], summing the expression for the c-th convolution of $\frac{\delta(r-vt)}{2\pi r}$. However, we defer the two dimensional case to the next section, where it will be obtained solving an integral equation.

The Bessel functions identities which allow to find the c-th convolution of an expanding spherical shell in 2 and 4 dimensions do not work in three dimensions [13].

6.5. Four Dimensions

ρ_s for four dimensions was obtained by Orsingher and De Gregorio [13]. The $(c+1)$-th space-time convolution of s in 4 dimensions is ([13], formula 1.5)

$$s^{\otimes(c+1)}(t,r) = \frac{1}{\pi^2(v^2t^2-r^2)v^2t^2}\frac{c+1}{(c-1)!}\left(\frac{v^2t^2-r^2}{v^2t}\right)^c. \tag{48}$$

Therefore [13]:

$$\rho_s(t,\vec{x}) = \sum_{c=0}^{\infty} e^{-\frac{t}{\tau}}\frac{\otimes^{c+1}s(t,\vec{x})}{\tau^c} = e^{-\frac{t}{\tau}}\ \frac{1}{\pi^2v^4t^3\tau}\exp\left(\frac{v^2t^2-r^2}{v^2t\tau}\right)\left[2+\frac{v^2t^2-r^2}{v^2t\tau}\right] =$$

$$\frac{1}{\pi^2v^4t^3\tau}\exp\left(-\frac{r^2}{v^2t\tau}\right)\left[2+\frac{v^2t^2-r^2}{v^2t\tau}\right]. \tag{49}$$

We shall see in section 10 that the four-dimensional random flight has applications to dimensions 1, 2 and 3.

6.6. Six Dimensions

ρ_s in six dimensions has been obtained by Kolesnik [21]. The space-time convolution of s with itself in 6 dimensions is ([21], formula (13))

$$s^{\otimes 2}(t,r) = \frac{16t}{\pi^3(vt)^6}\left(1-\frac{5}{6}\frac{r^2}{v^2t^2}\right). \tag{50}$$

When the number of collisions is greater than 1,

$$s^{\otimes c}(t,r) = \frac{(c+1)!t^c}{2\pi^3(vt)^6}\sum_{k=0}^{c+1}\frac{(k+1)(k+2)(c+2k+1)}{3^k(c-k+1)!(c+k-2)!}\,F\left(-(c+k-2),k+3;3;\frac{r^2}{v^2t^2}\right). \tag{51}$$

This expression is more friendly than it seems, because for the arguments above the hypergeometric function is actually a polynomial in $\frac{r^2}{v^2t^2}$. The Born expansion yields:

$$\rho_s(t,\vec{x}) = e^{-\frac{t}{\tau}}\frac{\delta(r-vt)}{\pi^3(vt)^5} + e^{-\frac{t}{\tau}}\frac{16t}{\tau\pi^3(vt)^6}\left(1-\frac{5}{6}\frac{r^2}{v^2t^2}\right)+$$

$$\frac{e^{-\frac{t}{\tau}}}{2\pi^3(vt)^6}\sum_{c=2}^{\infty}\left(\frac{t}{\tau}\right)^c(c+1)!\sum_{k=0}^{c+1}\frac{(k+1)(k+2)(c+2k+1)}{3^k(c-k+1)!(c+k-2)!}\,F\left(-(c+k-2),k+3;3;\frac{r^2}{v^2t^2}\right) \tag{52}$$

We shall see in section 10 that the six-dimensional random flight has applications to dimensions 1, 2 and 3.

The projections of the surface of the d-dimensional sphere on a d'-dimensional equatorial disk are (see, e. g., formula (59) in [22]):

$$\frac{\Gamma(d/2)(\sqrt{1-r^2})^{d-d'-2}}{\Gamma((d-d')/2)\pi^{d'/2}} \quad \text{for } r<1. \tag{53}$$

Comparison of the preceding formula with formula (48) and with the terms obtained when expanding the exponential in ρ_s for two dimensions (62) show that the probabilities of c collisions in 2 and 4 dimensions are projections of a constant density on the surface of higher dimensional spheres, as remarked by some authors [12, 13, 22]. This is not the case in 1 and 6 dimensions.

7. The Integral Equation

We define

$$\eta(t,\vec{x}) \equiv e^{-\frac{t}{\tau}} s(t,\vec{x}). \tag{54}$$

Then the integral equation

$$\rho_s(t,\vec{x}) = \eta(t,\vec{x}) + \frac{1}{\tau}(\eta \otimes \rho_s)(t,\vec{x}) = \eta(t,\vec{x}) + \frac{1}{\tau}\int_0^t dt' \int d\vec{x}'\, \eta(t',\vec{x}')\rho_s(t-t',\vec{x}-\vec{x}') \tag{55}$$

is the statement that at a given $(t,\vec{x})$ a particle has either scattered or not, and that in the second case it scattered for the first time at some $(t',\vec{x}')$ and at that place and time the process started again giving birth to ρ_s with origin in $(t',\vec{x}')$.

To undo the space convolution in the above equation we take the Fourier transform (to be denoted by $\tilde{}$) and to undo the time convolution we take the Laplace transform (to be denoted by $\hat{}$). The result is

$$\hat{\tilde{\rho}}_s = \hat{\tilde{\eta}} + \frac{1}{\tau}\hat{\tilde{\eta}}\hat{\tilde{\rho}}_s, \tag{56}$$

from which $\hat{\tilde{\rho}}_s$ can be solved:

$$\hat{\tilde{\rho}}_s = \frac{\hat{\tilde{\eta}}}{1-\hat{\tilde{\eta}}/\tau}. \tag{57}$$

The Fourier-Laplace inversion of $\hat{\tilde{\rho}}_s$ solves the problem. This approach was pioneered by Montroll and Weiss [23].

Or one can take the Fourier-Laplace on the Born expansion (29) of ρ_u:

$$\hat{\tilde{\rho}}_u(\omega,\vec{\nu}) = \frac{\hat{\tilde{u}}}{1-\frac{1}{\tau}\hat{\tilde{s}}}(\omega+\frac{1}{\tau},\vec{\nu}), \tag{58}$$

which is the above equation when $u=s$.

This section and the preceding one are computationally similar. In the previous section, in order to compute $s^{\otimes(c+1)}$, $\hat{\tilde{s}}$ was raised to the $(c+1)$-th power and Fourier-Laplace inverted in references [20] and [13] (see also (3.1) and (4.8) in [9] for a general expression of $\hat{\tilde{s}}^{c+1}$ in d dimensions). $s^{\otimes(c+1)}$ was then multiplied by $1/\tau^c$ and summed. In this section one sums the geometric series $\sum_{c=0}^{\infty}(1/\tau^c)\,\hat{\tilde{s}}^{c+1}(t,\vec{x})$ to obtain $\hat{\tilde{\rho}}_s$ and then Fourier-Laplace inverts it. Because of this similarity the one dimensional case using the integral equation will not be discussed.

For the bullet initial condition case there is an integral equation similar to the second one of this section which will be presented in the subsection that follows.

7.1. Two Dimensions

The definitions of Fourier and Laplace transform that we use are:

$$\tilde{f}(\nu) \equiv \int_{-\infty}^{\infty} dx\; f(x)\, e^{-i2\pi\nu x} \quad \text{and} \quad \hat{f}(\omega) \equiv \int_{0}^{\infty} dt\; f(t)\, e^{-\omega t}. \tag{59}$$

In two dimensions the Fourier-Laplace transform of η is:

$$\hat{\tilde{\eta}}(\omega,\vec{\nu}) = \frac{1}{\sqrt{(\frac{1}{\tau}+\omega)^2 + (2\pi v\nu)^2}} \tag{60}$$

and, according to formula (57),

$$\hat{\tilde{\rho}}_s(\omega,\vec{\nu}) = \frac{1}{\sqrt{(1/\tau+\omega)^2 + (2\pi v\nu)^2} - 1/\tau}. \tag{61}$$

This expression was inverted by Masoliver *et alii* [24] to yield:

$$\rho_s(t,\vec{x}) = e^{-\frac{t}{\tau}}\left[\frac{\delta(r-vt)}{2\pi r} + \frac{1}{2\pi v\tau\sqrt{(vt)^2-r^2}}\exp\left(\frac{1}{v\tau}\sqrt{(vt)^2-r^2}\right)H(vt-r)\right], \tag{62}$$

where H is the Heaviside function.

The solution for the bullet initial condition along the positive direction of the x axis can be obtained from the formula

$$\rho_b(t,\vec{x}) = e^{-\frac{t}{\tau}}\delta(vt-x)\delta(y) + \frac{1}{\tau}\int_0^t dt'\; e^{-\frac{t'}{\tau}}\rho_s(t-t', |\vec{x}-vt'\vec{i}|). \tag{63}$$

It is the statement that at a given $(t,\vec{x})$ a particle has either scattered or not, and that in the second case it scattered for the first time at some $(t',\vec{x}')$ and at that place and time it gave birth to ρ_s with origin in $(t',\vec{x}')$. The last formula is also equivalent to formula (30) when the initial condition u is the bullet initial condition b. Substitution of ρ_s (expression (62)) into the previous formula yields [25, 8]

$$\rho_b(t,\vec{x}) = e^{-\frac{t}{\tau}}\left[\delta(vt-x)\delta(y) + \frac{1}{2\pi v\tau(vt-x)}\exp\left(\frac{1}{v\tau}\sqrt{(vt)^2-r^2}\right)H(vt-r)\right]. \tag{64}$$

7.2. Three Dimensions

In three dimensions the Fourier-Laplace transform of η is [25]:

$$\hat{\tilde{\eta}}(\omega,\vec{\nu}) = \frac{1}{2\pi v\nu}\ \mathrm{arc\ tg}\frac{2\pi v\nu}{(1/\tau)+\omega}. \tag{65}$$

and, according to formula (57) [25, 26],

$$\hat{\tilde{\rho}}_s(\omega,\vec{\nu}) = \frac{\mathrm{arc\ tg}\frac{2\pi v\nu}{(1/\tau)+\omega}}{2\pi v\nu - \frac{1}{\tau}\ \mathrm{arc\ tg}\frac{2\pi v\nu}{(1/\tau)+\omega}} \tag{66}$$

is the Fourier-Laplace transform of the solution for isotropic initial conditions.

To obtain the Fourier-Laplace transform of the solution for the bullet initial condition we use equation (58). The bullet initial condition in 3 dimensions is

$$b_x(t,\vec{x}) = \delta(vt-x)\delta(y)\delta(z) \tag{67}$$

and its Fourier-Laplace transform is

$$\hat{\tilde{b}}_x(\omega,\vec{\nu}) = \frac{1}{\omega + i2\pi v\nu_x}. \tag{68}$$

Equation (58) yields [25, 8]

$$\hat{\tilde{\rho}}_x(\omega,\vec{\nu}) = \frac{1}{\frac{1}{\tau}+\omega+i2\pi v\nu_x}\ \frac{2\pi v\nu}{2\pi v\nu - \frac{1}{\tau}\ \mathrm{arc\ tg}\frac{2\pi v\nu}{(1/\tau)+\omega}}. \tag{69}$$

No analytic Fourier-Laplace inversions of expressions (66) and (69) are known.

8. The Differential Equation

8.1. One Dimension

The natural way to obtain a differential equation for the basic process is to define a discrete space-time version of the model, obtain a finite difference equation and then take the continuum limit of this finite difference equation. This was the first approach to the problem, and was done in 1 dimension by Goldstein [27].

Regarding the problem as a random flight rather than as a persistent random walk leads to a simpler derivation of the differential equation in 1 dimension because it does not require to define densities of left and right moving particles as in [27]. Let Δt and $v\ \Delta t$ denote the time and space steps, respectively. Let $(\Delta t)/\tau$ be the probability that the particle has of diffusing isotropically at each site, where τ has dimensions of time. The mean time between two collisions is then

$$\frac{(\Delta t)}{\tau}(\Delta t) + \left(1-\frac{(\Delta t)}{\tau}\right)\frac{(\Delta t)}{\tau}(2\Delta t) + \cdots + \left(1-\frac{(\Delta t)}{\tau}\right)^{j-1}\frac{(\Delta t)}{\tau}(j\Delta t) + \cdots =$$

$$\frac{(\Delta t)^2}{\tau}\sum_{j=0}^{\infty} j\left(1-\frac{(\Delta t)}{\tau}\right)^{j-1} = \frac{(\Delta t)^2}{\tau}\frac{d}{dx}\;\frac{1}{1-x}\bigg|_{x=1-\frac{(\Delta t)}{\tau}} =$$

$$\frac{(\Delta t)^2}{\tau}\frac{1}{\left(1-\left(1-\frac{(\Delta t)}{\tau}\right)\right)^2} = \tau. \tag{70}$$

There are two contributions to the particles that arrive at time $t+\Delta t$ at site x. There are particles which have been scattered at the neighbouring sites and particles that have not. The first contribution is $(\Delta t/2\tau)(\rho(t,x-v\Delta t)+\rho(t,x+v\Delta t))$ (the factor 1/2 is there because half of the particles are scattered to the sites $x\pm 2v\Delta t$). The second contribution is not $(1-(\Delta t)/\tau)(\rho(t,x-v\Delta t)+\rho(t,x+v\Delta t))$ because one has to discount the particles that arrived to site $x-v\Delta t$ from the right and to site $x+v\Delta t$ from the left. But these are precisely the particles that were at site x at time $t-\Delta t$. Therefore the second contribution is $(1-(\Delta t)/\tau)(\rho(t,x-v\Delta t)+\rho(t,x+v\Delta t)-\rho(t-\Delta t,0))$, and the equation is

$$\rho(t+\Delta t,x) =$$

$$\frac{\Delta t}{2\tau}\;(\rho(t,x-v\Delta t)+\rho(t,x+v\Delta t)) + \left(1-\frac{\Delta t}{\tau}\right)(\rho(t,x-v\Delta t)+\rho(t,x+v\Delta t)-\rho(t-\Delta t,0))$$

$$= \left(1-\frac{\Delta t}{2\tau}\right)(\rho(t,x-v\Delta t)+\rho(t,x+v\Delta t)) + (\frac{1}{\tau}\Delta t-1)\rho(t-\Delta t,0). \tag{71}$$

The Taylor expansions up to second order around $\rho(t,x)$,

$$\rho(t+\Delta t,x) = \rho(t,x) + \frac{\partial\rho}{\partial t}(t,x)\Delta t + \frac{1}{2}\frac{\partial^2\rho}{\partial t^2}(t,x)(\Delta t)^2 + o((\Delta t)^2)$$

and

$$\rho(t,x+v\Delta t) = \rho(t,x) + \frac{\partial\rho}{\partial x}(t,x)v\Delta t + \frac{1}{2}\frac{\partial^2\rho}{\partial x^2}(t,x)v^2(\Delta t)^2 + o((\Delta t)^2). \tag{72}$$

yield the differential equation

$$\frac{\partial^2\rho}{\partial t^2} + \frac{2/\tau}{2-(\Delta t)/\tau}\frac{\partial\rho}{\partial t} = v^2\frac{\partial^2\rho}{\partial x^2} + \frac{o((\Delta t)^2)}{(\Delta t)^2}. \tag{73}$$

For any Δt the above process has τ as its mean time between collisions. In particular, when $\Delta t\to 0$, the resulting process is a Poisson process of the sort that we are interested in and whose solution is the solution of the telegrapher's equation

$$\frac{\partial^2\rho}{\partial t^2} + \frac{1}{\tau}\frac{\partial\rho}{\partial t} = v^2\frac{\partial^2\rho}{\partial x^2}. \tag{74}$$

This equation has a long history dating back to Lord Kelvin and the laying of the first submarine transatlantic cable [28]. We show how to solve it in Appendix II, following Webster [28]. In this section we give the solutions without derivation.

The solution of the telegrapher's equation with initial conditions

$$\rho(0,x)=\delta(x) \quad \text{and} \quad \frac{\partial\rho}{\partial t}(0,x)=0, \tag{75}$$

is

$$\rho_s(t,x)=\frac{1}{2}\,e^{-\frac{t}{2\tau}}\left[\delta(x-vt)+\delta(x+vt)\right]+ \tag{76}$$

$$\frac{1}{2}\,e^{-\frac{t}{2\tau}}\left[\frac{1}{v}\frac{1}{2\tau}\,I_0\left(\frac{1}{2\tau}\sqrt{t^2-\frac{x^2}{v^2}}\right)+\left(\frac{1}{2\tau}\right)^2\frac{t}{v}\,\frac{I_1\left(\frac{1}{2\tau}\sqrt{t^2-\frac{x^2}{v^2}}\right)}{\frac{1}{2\tau}\sqrt{t^2-\frac{x^2}{v^2}}}\right]H(vt-|x|),$$

where I_0 and I_1 are the Bessel functions of imaginary argument,

$$I_0(z)=\sum_{j=0}^{\infty}\frac{(z/2)^{2j}}{(j!)^2}, \qquad I_1(z)=\sum_{j=0}^{\infty}\frac{(z/2)^{2j+1}}{j!(j+1)!}. \tag{77}$$

There is a prefactor of $e^{-\frac{t}{2\tau}}$ as opposed to the prefactor $e^{-\frac{t}{\tau}}$ in the Born expansion. This is a feature of one dimension only, and its explanation is the following. In one dimension the statements "meeting a scatterer with probability dt/τ and then scattering isotropically" and "meeting a scatterer with probability $dt/2\tau$ and then scattering backwards" are equivalent. This will be shown formally in the section "Anisotropic scattering".

When the initial condition is the bullet initial condition,

$$\rho(0,x)=\delta(x) \quad \text{and} \quad \frac{\partial\rho}{\partial t}(0,x)=\left.\frac{\partial\delta(x-vt)}{\partial t}\right|_{t=0}=-v\delta'(x), \tag{78}$$

the solution is

$$\rho_b(t,x)=e^{-\frac{t}{2\tau}}\,\delta(x-vt)+$$

$$\frac{1}{2}\,e^{-\frac{t}{2\tau}}\left[\frac{1}{v}\frac{1}{2\tau}\,I_0\left(\frac{1}{2\tau}\sqrt{t^2-\frac{x^2}{v^2}}\right)+\left(\frac{1}{2\tau}\right)^2\frac{vt+x}{v^2}\,\frac{I_1\left(\frac{1}{2\tau}\sqrt{t^2-\frac{x^2}{v^2}}\right)}{\frac{1}{2\tau}\sqrt{t^2-\frac{x^2}{v^2}}}\right]H(vt-|x|). \tag{79}$$

8.2. Higher Dimensions

It is tempting to think that the solution of the telegrapher's equation in d dimensions,

$$\frac{\partial^2\rho}{\partial t^2} + \frac{1}{\tau}\frac{\partial\rho}{\partial t} = v^2\nabla^2\rho, \tag{80}$$

with appropriate initial conditions is the non singular part of the function $\rho(t,x)$ which solves our problem, but this is true only in 1 dimension.

A differential equation for the two-dimensional case has been obtained essentially by transforming the product by ω in the Laplace variable by a derivative with respect to time in the time variable [24]. To do this rewrite equation (61) as

$$(\omega^2 + 2\frac{\omega}{\tau} + (2\pi v\nu)^2)\hat{\tilde{\rho}}_s(\omega,\vec{\nu}) = \frac{1}{\tau} + \big((\frac{1}{\tau}+\omega)^2 + (2\pi v\nu)^2\big)\hat{\tilde{\eta}}_s(\omega,\vec{\nu}). \tag{81}$$

The Laplace transform (denoted momentarily by $\pounds$) of the derivative of a function satisfies the following identities:

$$\left(\pounds\frac{df}{dt}\right)(\omega) = \omega\left(\pounds f\right)(\omega) - f(0),$$

$$\left(\pounds\frac{d^2f}{dt^2}\right)(\omega) = \omega^2\left(\pounds f\right)(\omega) - \omega f(0) - \frac{df}{dt}(0). \tag{82}$$

Substituting the identities into their preceding equation yields (the first term in the rhs is missing in [24])

$$\frac{\partial^2\rho}{\partial t^2} + \frac{2}{\tau}\frac{\partial\rho}{\partial t} - v^2\nabla^2\rho = \frac{1}{\tau}\,\delta(t)\frac{\delta(r)}{2\pi r} + \frac{v^2}{r}\,e^{-\frac{t}{\tau}}\,\frac{\partial}{\partial r}\left(\frac{\delta(r-vt)}{2\pi r}\right). \tag{83}$$

Both the solution (62) and the solution (64) to the two-dimensional problem satisfy the above differential equation when $r\in(0,vt)$, but not at the extrema of the interval. They satisfy the equation

$$\frac{\partial^2\rho}{\partial t^2} + \frac{2}{\tau}\frac{\partial\rho}{\partial t} - v^2\nabla^2\rho = 0. \tag{84}$$

when $r\in[0,vt)$. Note that this equation is a telegrapher's equation, but the coefficient of the first order term in t is $\frac{2}{\tau}$, not $\frac{1}{\tau}$.

Applying the above procedure to the Fourier-Laplace transform of the solution in three dimensions (66) yields a partial differential equation of infinite order, because of the series expansion of $\operatorname{arc\,tg}$. The first two terms of this expansion are a telegrapher's equation [24]. This procedure has been applied to arbitrary dimension and studied extensively [29].

9. Anisotropic Scattering

It is important to distinguish the problem of random flight with drift (or bias as it is also called in 1 dimension) from the problem of random flight with an anisotropic angular cross section. In both cases each scattering gives birth to some expanding probability cloud

$f(t,\vec{x})$ which is not isotropic. In the first case the orientation of this cloud is constant, and for a given initial condition u the Born expansion of the solution would be

$$\rho_u(t,\vec{x}) = e^{-\frac{t}{\tau}} \left(\sum_{c=0}^{\infty} \left(\frac{f \otimes}{\tau} \right)^c u \right) (t,\vec{x}). \tag{85}$$

This case has also been called asymmetric random walk ([30], page 107) or non-symmetrical random motion ([9], section 6). A physical model for this random flight with drift (similar to model number 1) would be provided by particles which scatter at random instants according to some *isotropic* angular cross section in the presence of some constant external force (say, the weight) which pulls them along a given direction.

The random walk with drift is completely solvable in 1 dimension, where one has to solve a differential equation which has a derivative with respect to x.

In the second case the orientation of the expanding probability cloud depends on the impinging direction and the previous expansion is not valid any more. As said earlier, not much has been achieved when the cross section is anisotropic, except for the 1 dimensional case, which can be solved, as we show now.

In the one-dimensional model the density of scatterers is $\frac{1}{v\tau}$. When the scattering is anisotropic, once a scatterer is met, there is a probability p of forward scattering and a probability q of backward scattering. But forward scattering is like not meeting any scatterer at all, so this situation is undistinguishable from a density of scatterers $\frac{q}{v\tau}$ and a probability 1 of backward scattering. Thus the triples $(p, q, \frac{1}{v\tau})$ and $(0, 1, \frac{q}{v\tau})$ lead to the same $\rho(t, x)$. More generally, the triples

$$\left\{ \left(1 - q\alpha, q\alpha, \frac{1}{v\tau\alpha} \right), \alpha \in \left(0, \frac{1}{q} \right] \right\} \tag{86}$$

all lead to the same $\rho(t, x)$. In particular $(p, q, \frac{1}{v\tau})$ and $(1/2, 1/2, \frac{2q}{v\tau})$ are equivalent. Therefore, when the mean free path is $v\tau$ and the isotropic scattering is given by the probabilities (p, q), $\rho(t, x)$ is the solution of the telegrapher's equation

$$\frac{\partial^2 \rho}{\partial t^2} + \frac{2q}{\tau}\frac{\partial \rho}{\partial t} = v^2 \frac{\partial^2 \rho}{\partial x^2}. \tag{87}$$

The prefactor 1/2 is needed because, as remarked after formula (6), the density is spread by a factor of two with respect to the binomial. After a quick numerical exploration it seems that the approximation to the binomial numbers furnished by this solution of this telegrapher's equation is better than the one furnished by the Gaussian for the central values, but worse for the tails.

When the dimension is 2 or more, the above trick will not do, except in the case in which the anisotropic cross section is simply a continuous anisotropic part and a singular forward or backward scattering component. One way to study the anisotropic case is to observe that although the process is not Markovian in the positions, it is Markovian in the space of the positions and the directions. An integro-differential equation for the joint density of the positions and the directions has been obtained [25]. Another approach is to substitute the space convolution by a more complicated convolution which rotates the cross section according to the impinging direction. This has been done in references [24] (section 3) and [8] (sections 2 and 5). In the first reference an example in two dimensions was studied and the time evolution of its second moment found, in the second reference techniques to approximate $\hat{\hat{\rho}}$ have been developed in two and three dimensions.

10. Projections onto Lower Dimensional Spaces

The projection of a random flight onto an Euclidean subspace is a generalized random flight. This follows from the fact that the convolution of two probability densities and their projection onto an Euclidean subspace commute. This is straightforward to check. Let f and g be two functions of x and y, and let P_y denote the operator which projects along the y axis. Then:

$$P_y(f \otimes_s g)(x) = \int dy \left(\int dx' \int dy' f(x', y') g(x - x', y - y') \right) =$$

$$\int dx' \left(\int dy' f(x', y') \right) \left(\int d(y - y') g(x - x', y - y') \right) = (P_y f \otimes_s P_y g)(x). \quad (88)$$

The support of the functions are extended to all of R^2, letting them be zero wherever necessary. Then the integrals go from $-\infty$ to $+\infty$, and the above change from dy to $d(y - y')$ is allowed. This commutation also holds for a larger number of variables or projections or functions, so that it can be applied to the Born expansion:

$$(P\rho_u)(t, \vec{x}) = e^{-\frac{t}{\tau}} \left(\sum_{c=0}^{\infty} \left(\frac{Ps\, \otimes}{\tau} \right)^c Pu \right)(t, \vec{x}), \quad (89)$$

where P is the projection operator over some Euclidean subspace of R^d.

This shows that the projection of a random flight with initial condition u onto an Euclidean subspace $R^{d'}$ $(d' < d)$ is a random walk with initial condition Pu which differs from a random flight only by the fact that after each scattering the expanding probability cloud is not an expanding spherical shell s but the projection Ps of it. As s, Ps is isotropic. In contrast with s, where the particle has speed v after the scattering, in Ps all speeds between 0 and v are possible.

Thus by projecting the 4-dimensional solution (49) we obtain solutions for random walks on 3, 2 and 1 dimensions, and by projecting the 2-dimensional solution (62) we obtain solutions for a random walk on 1 dimension. These projections and the corresponding projections of the spheres have been computed by Orsingher and De Gregorio ([13], section 4, but factor of 2 is missing in equations (96) and (94) in reference [13]).

Projections of the 4-dimensional random flight:
Projection on R^3:

$$(Ps)(t, \vec{x}) = \frac{1}{(\pi v t)^2} \frac{1}{\sqrt{(vt)^2 - r^2}} \quad \text{for } r < vt. \quad (90)$$

$$(P\rho_s)(t,\vec{x}) = e^{-\frac{t}{\tau}} \frac{\sqrt{\pi/\tau}}{\pi^2 v^3 t^{5/2}} \sum_{c=0}^{\infty} \frac{k+1}{\Gamma\big(k+1/2\big)} \left(\frac{v^2t^2 - r^2}{v^2\tau t}\right)^{k-\frac{1}{2}} \quad \text{for } r < vt. \tag{91}$$

Projection on R^2:

$$(Ps)(t,\vec{x}) = \frac{1}{\pi(vt)^2} \quad \text{for } r < vt. \tag{92}$$

$$(P\rho_s)(t,\vec{x}) = \frac{e^{-\frac{r^2}{v^2\tau t}}}{\pi(vt)^2} \left(1 + \frac{v^2t^2 - r^2}{v^2\tau t}\right) \quad \text{for } r < vt. \tag{93}$$

Projection on R:

$$(Ps)(t,x) = \frac{2}{\pi(vt)^2}\sqrt{(vt)^2 - x^2} \quad \text{for } |x| < vt. \tag{94}$$

$$(P\rho_s)(t,x) = \frac{\sqrt{\tau}e^{-\frac{t}{\tau}}}{v\sqrt{\pi t^3}} \sum_{c=0}^{\infty} \frac{k+1}{\Gamma\big(k+3/2\big)} \left(\frac{v^2t^2 - x^2}{v^2\tau t}\right)^{k+\frac{1}{2}} \quad \text{for } |x| < vt. \tag{95}$$

Projection of the 2-dimensional random flight:
Projection on R:

$$(Ps)(t,x) = \frac{1}{\pi\sqrt{(vt)^2 - x^2}} \quad \text{for } |x| < vt. \tag{96}$$

$$(P\rho_s)(t,x) = \frac{e^{-\frac{t}{\tau}}}{2v\tau} \sum_{c=0}^{\infty} \frac{1}{\Gamma^2\big((k+1)/2\big)} \left(\frac{\sqrt{(vt)^2 - x^2}}{2v\tau}\right)^{k-1} \quad \text{for } |x| < vt. \tag{97}$$

The projections of the six-dimensional random flight [21] have not been computed. The projections of the surface of the six-dimensional sphere on a d'-dimensional equatorial disk are (from (53)):

$$\frac{2(\sqrt{1-r^2})^{6-d'-2}}{\Gamma((6-d')/2)\pi^{d'/2}} \quad \text{for } r < 1. \tag{98}$$

For research related to this section see [22].

11. Some Other Topics

The present chapter is a review of the essentials of the persistent random walk. In this section we hopefully mention the most important areas that one would need to cover to have a general knowledge of the persistent random walk and we make comments on some of them.

The persistent random walk in lattices can be developed according to the scheme presented in this chapter, simply letting $s(t, \vec{x})$ be an appropriate weighed sum of Dirac deltas. When the lattice is cubic, the expression for it is given in (10). The persistent random walk in a 1-dimensional lattice was solved by Goldstein [27], who solved the difference equation (71). The biased persistent random walk in a 1-dimensional lattice was solved recently [31], not by solving a difference equation but using combinatorial methods. In cubic lattices of dimension higher than 1 the persistent random walk has been studied by Halpern [32], who has obtained results on the Fourier and Laplace transforms of ρ, as well as on its second moment. It has been shown that the telegrapher's equation in more than one dimension does not describe the persistent random walk in cubic lattices, either [33, 18].

From a physical point of view model 1 (sections 1 and 4) requires the scatterers to be of infinite mass, so that they remain still during the scattering and do not change the energy (speed) of the scattered particle. These conditions are met by the diffusion of photons [34], because the speed of the center of mass frame in this scattering is negligible compared to the speed of light. When the photons scatter in a turbid medium [35, 36], [37] (where the anisotropic scattering is applied) then the scatterers are moving, as in model 4.

The persistent random walk is a model of thermophoresis [38]. When it takes place in a lattice it is a model of electrical conductivity [39]. There are persistent random walks with reflecting walls [40] and traps [36]. Their first passage times have also been studied [41]. The persistent random walk also models processes in biology [42, 43]. For more applications see [2].

12. Appendix I: Computation of the Sixth Moment

First, we compute $\sigma(3, d, c)$. The partitions of 3 are $\{3\}, \{1, 2\}$ and $\{1, 1, 1\}$. These can be realized by $c+1$ terms in $c+1$, $c(c+1)$ and $\binom{c+1}{3}$ ways, respectively. Therefore, from the definition (30) of $\sigma(m, d, c)$:

$$\sigma(3, d, c) = \frac{\Gamma(3+d/2)}{\Gamma(3+1/2)} \sum_{\substack{i_1,\cdots,i_{c+1}\in N \\ i_1+\cdots+i_{c+1}=3}} \frac{\Gamma(i_1+1/2)\cdots\Gamma(i_{c+1}+1/2)}{\Gamma(i_1+d/2)\cdots\Gamma(i_{c+1}+d/2)} =$$

$$\frac{\Gamma(3+d/2)}{\Gamma(3+1/2)}\Big[(c+1)\frac{\Gamma(3+1/2)\Gamma(1/2)^c}{\Gamma(3+d/2)\Gamma(d/2)^c}+$$

$$c(c+1)\frac{\Gamma(2+1/2)\Gamma(1+1/2)\Gamma(1/2)^{c-1}}{\Gamma(2+d/2)\Gamma(1+d/2)\Gamma(d/2)^{c-1}} + \binom{c+1}{3}\frac{\Gamma(1+1/2)^3\Gamma(1/2)^{c-2}}{\Gamma(1+d/2)^3\Gamma(d/2)^{c-2}}\Big]. \quad (99)$$

Using the $\Gamma(z+1) = z\Gamma(z)$ property of the Γ function:

$$\sigma(3, d, c) = \left(c+1+c(c+1)\frac{4+d}{5d} + \frac{(c+1)c(c-1)}{90}\frac{(4+d)(2+d)}{d^2}\right)\left(\frac{\Gamma(1/2)}{\Gamma(d/2)}\right)^c \quad (100)$$

Substitution of this result into the formula (18) for the even radial moments yields

$$\langle r^6 \rangle =$$

$$6!\, e^{-\frac{t}{\tau}}\, (vt)^6 \sum_{c=0}^{\infty} \frac{(t/\tau)^c}{(6+c)!} \left(c+1+c(c+1)\frac{4+d}{5d} + \frac{(c+1)c(c-1)}{90}\frac{(4+d)(2+d)}{d^2} \right) =$$

$$6!\, e^{-\frac{t}{\tau}}\, (v\tau)^6 \sum_{c=6}^{\infty} \frac{(t/\tau)^c}{c!} \left(c-5+(c-6)(c-5)\frac{4+d}{5d} + \frac{(c-5)(c-6)(c-7)}{90}\frac{(4+d)(2+d)}{d^2} \right) = \tag{101}$$

$$6!\, e^{-\frac{t}{\tau}}\, (v\tau)^6 \sum_{c=0}^{\infty} \frac{(t/\tau)^c}{c!} \left(c-5+(c-6)(c-5)\frac{4+d}{5d} + \frac{(c-5)(c-6)(c-7)}{90}\frac{(4+d)(2+d)}{d^2} \right) -$$

$$6!\, e^{-\frac{t}{\tau}}\, (v\tau)^6 \sum_{c=0}^{5} \frac{(t/\tau)^c}{c!} \left(c-5+(c-6)(c-5)\frac{4+d}{5d} + \frac{(c-5)(c-6)(c-7)}{90}\frac{(4+d)(2+d)}{d^2} \right) = \tag{102}$$

$$6!\, e^{-\frac{t}{\tau}}\, (v\tau)^6 \left(\frac{t}{\tau}\, e^{\frac{t}{\tau}} - 5e^{\frac{t}{\tau}} \right) +$$

$$6!\, e^{-\frac{t}{\tau}}\, (v\tau)^6 \sum_{c=0}^{\infty} \frac{(t/\tau)^c}{c!} \left((c(c-1)-10c+30)\frac{4+d}{5d} \right) +$$

$$6!\, e^{-\frac{t}{\tau}}\, (v\tau)^6 \sum_{c=0}^{\infty} \frac{(t/\tau)^c}{c!} \left(\frac{c(c-1)(c-2)-15c(c-1)+90c-210}{90}\frac{(4+d)(2+d)}{d^2} \right) -$$

$$6!\, e^{-\frac{t}{\tau}}\, (v\tau)^6 \sum_{c=0}^{5} \frac{(t/\tau)^c}{c!} \left(c-5+(c-6)(c-5)\frac{4+d}{5d} + \frac{(c-5)(c-6)(c-7)}{90}\frac{(4+d)(2+d)}{d^2} \right) = \tag{103}$$

$$6!\, e^{-\frac{t}{\tau}}\, (v\tau)^6 \left[\frac{t}{\tau} e^{\frac{t}{\tau}} - 5e^{\frac{t}{\tau}} + \frac{4+d}{5d}\left(\frac{t^2}{\tau^2}\, e^{\frac{t}{\tau}} - 10\frac{t}{\tau}\, e^{\frac{t}{\tau}} + 30e^{\frac{t}{\tau}} \right) + \right.$$

$$\left. \frac{(4+d)(2+d)}{90d^2}\left(\frac{t^3}{\tau^3}\, e^{\frac{t}{\tau}} - 15\frac{t^2}{\tau^2}\, e^{\frac{t}{\tau}} + 90\frac{t}{\tau}\, e^{\frac{t}{\tau}} - 210e^{\frac{t}{\tau}} \right) \right] -$$

$$6!\, e^{-\frac{t}{\tau}}\, (v\tau)^6 \sum_{c=0}^{5} \frac{(t/\tau)^c}{c!} \left[c-5+(c-6)(c-5)\frac{4+d}{5d} + \frac{(c-5)(c-6)(c-7)}{90}\frac{(4+d)(2+d)}{d^2} \right] = \tag{104}$$

$$6!\, (v\tau)^6 \left[\frac{t}{\tau} - 5 + \frac{4+d}{5d}\left(\frac{t^2}{\tau^2} - 10\frac{t}{\tau} + 30 \right) + \frac{(4+d)(2+d)}{90d^2}\left(\frac{t^3}{\tau^3} - 15\frac{t^2}{\tau^2} + 90\frac{t}{\tau} - 210 \right) \right] -$$

$$6!\, e^{-\frac{t}{\tau}}\, (v\tau)^6 \sum_{c=0}^{5} \frac{(t/\tau)^c}{c!} \left[c-5+(c-6)(c-5)\frac{4+d}{5d} + \frac{(c-5)(c-6)(c-7)}{90}\frac{(4+d)(2+d)}{d^2} \right]. \tag{105}$$

13. Appendix II: Solution of the Telegrapher's Equation

13.1. Change of Variables

Consider the equation

$$\left(A\frac{\partial\rho}{\partial t} + B\frac{\partial^2\rho}{\partial t^2} - C\frac{\partial\rho}{\partial x} - D\frac{\partial^2\rho}{\partial x^2} + E\right)\rho(t,x) = 0, \tag{106}$$

where A, B, C, D, E are constants, and $B > 0, D > 0$. First we complete the square in each differential operator:

$$\left[B\left(\frac{\partial}{\partial t} + \frac{A}{2B}\right)^2 - D\left(\frac{\partial}{\partial x} + \frac{C}{2D}\right)^2 + \left(E - \frac{A^2}{4B} + \frac{C^2}{4D}\right)\right]\rho(t,x) = 0. \tag{107}$$

We define φ by

$$\rho(t,x) = \exp\left(-\frac{At}{2B}\right)\ \exp\left(-\frac{Cx}{2D}\right)\ \varphi(t,x). \tag{108}$$

Since $\frac{A}{2B}$ cancels the derivative that comes from applying Leibniz's rule to $\rho(t,x)$,

$$\left(\frac{\partial}{\partial t} + \frac{A}{2B}\right)^n \exp\left(-\frac{At}{2B}\right)\ \varphi(t,x) = \exp\left(-\frac{At}{2B}\right)\ \frac{\partial^n}{\partial t^n}\,\varphi(t,x), \tag{109}$$

and the same thing happens for the x variable, the differential equation in terms of the new unknown φ is the Klein-Gordon equation,

$$\left(B\frac{\partial^2}{\partial t^2} - D\frac{\partial^2}{\partial x^2}\right)\varphi(t,x) = \left(\frac{A^2}{4B} - \frac{C^2}{4D} - E\right)\varphi(t,x). \tag{110}$$

Suppose now that $\frac{A^2}{2B} - \frac{C^2}{2D} - E > 0$ (otherwise we move the rhs to the lhs so that φ has a positive coefficient). Then this rescaling of the variables and redefinition of the unknown

$$\kappa \equiv \alpha\, t,\ \ u \equiv \beta\, x,\ \ \phi(\kappa,u) \equiv \varphi(t(\kappa), x(u)), \tag{111}$$

where

$$\alpha \equiv \sqrt{\frac{\frac{A^2}{4B} - \frac{C^2}{4D} - E}{B}},\ \ \beta \equiv \sqrt{\frac{\frac{A^2}{4B} - \frac{C^2}{4D} - E}{D}} \tag{112}$$

turns all three coefficients into the same one:

$$\left(\frac{\partial^2}{\partial\kappa^2} - \frac{\partial^2}{\partial u^2}\right)\phi(\kappa,u) = \phi(\kappa,u), \tag{113}$$

13.2. Solution of the Canonical Form

If we define the Fourier transform of ϕ by:

$$\phi(\kappa,u)=\int_{-\infty}^{+\infty}d\nu\;\tilde{\phi}(\kappa,\nu)\,e^{i2\pi\nu u}, \tag{114}$$

then the last differential equation yields the harmonic oscillator equation for $\tilde{\phi}(\kappa,\nu)$:

$$\frac{\partial^2}{\partial\kappa^2}\,\tilde{\phi}(\kappa,\nu)=\left(1-4\pi^2\nu^2\right)\tilde{\phi}(\kappa,\nu),\quad \forall\nu\in R. \tag{115}$$

For initial conditions $\tilde{\phi}(0,\nu)$ and $\frac{\partial\tilde{\phi}}{\partial\kappa}(0,\nu)$ the solution to the above equation is:

$$\tilde{\phi}(\kappa,\nu)=\tilde{\phi}(0,\nu)\cos\left(\sqrt{4\pi^2\nu^2-1}\;\kappa\right)+\frac{\frac{\partial\tilde{\phi}}{\partial\kappa}(0,\nu)}{\sqrt{4\pi^2\nu^2-1}}\sin\left(\sqrt{4\pi^2\nu^2-1}\;\kappa\right). \tag{116}$$

Since

$$\tilde{\phi}(0,\nu)=\int_{-\infty}^{+\infty}du\;\phi(0,u)\,e^{-i2\pi\nu u}\quad\text{and}\quad\frac{\partial\tilde{\phi}}{\partial\kappa}(0,\nu)=\int_{-\infty}^{+\infty}du\;\frac{\partial\phi}{\partial\kappa}(0,u)\,e^{-i2\pi\nu u}, \tag{117}$$

it follows that

$$\phi(\kappa,u)=\int_{-\infty}^{+\infty}d\nu\;\int_{-\infty}^{+\infty}du'$$

$$\left[\phi(0,u')\;\cos\left(\sqrt{4\pi^2\nu^2-1}\;\kappa\right)+\frac{\frac{\partial\phi}{\partial\kappa}(0,u')}{\sqrt{4\pi^2\nu^2-1}}\;\sin\left(\sqrt{4\pi^2\nu^2-1}\;\kappa\right)\right]\;e^{i2\pi\nu(u-u')} \tag{118}$$

where the integration over ν undoes the Fourier transform of ϕ (see (113)) and the integration over u undoes the Fourier transform of its initial conditions (see (116)).

To compute the above integral the following identity (derived in [28], pp. 175-177) involving the order zero Bessel function, J, may be used:

$$\frac{\sin\left(\sqrt{4\pi^2\nu^2-1}\;\kappa\right)}{\sqrt{4\pi^2\nu^2-1}}=\frac{1}{2}\int_{-\kappa}^{\kappa}d\kappa'\;I_0\left(\sqrt{\kappa^2-\kappa'^2}\right)e^{-i2\pi\nu\kappa'}, \tag{119}$$

where $I_0(x)\equiv J(ix)$ is the Bessel function of imaginary argument, whose series expansion is:

$$I_0(z)=\sum_{j=0}^{\infty}\frac{(z/2)^{2j}}{(j!)^2}. \tag{120}$$

To do the integral of the second term we substitute this identity in the expression for $\phi(\kappa,u)$ and see that all the ν dependence is in the exponential $e^{i2\pi\nu(u-u'-\kappa')}$. Its integration over ν yields $\delta(u-u'-\kappa')$. Integration again over κ' yields

$$\frac{1}{2}\int_{u-\kappa}^{u+\kappa} du' \, \frac{\partial \phi}{\partial \kappa}(0,u') \, I_0\left(\sqrt{\kappa^2-(u-u')^2}\right). \tag{121}$$

Since $\frac{\partial}{\partial\kappa}\frac{\sin\left(\sqrt{4\pi^2\nu^2-1}\,\kappa\right)}{\sqrt{4\pi^2\nu^2-1}} = \cos\left(\sqrt{4\pi^2\nu^2-1}\,\kappa\right)$, the derivative with respect to κ of

$$\frac{1}{2}\int_{u-\kappa}^{u+\kappa} du' \, \phi(0,u') \, I_0\left(\sqrt{\kappa^2-(u-u')^2}\right) \tag{122}$$

yields the integral of the first term:

$$\frac{1}{2}\left[\phi(0,u-\kappa)+\phi(0,u+\kappa)\right] + \frac{1}{2}\int_{u-\kappa}^{u+\kappa} du' \, \phi(0,u') \, \frac{\partial}{\partial\kappa} I_0\left(\sqrt{\kappa^2-(u-u')^2}\right). \tag{123}$$

Thus the solution is:

$$\phi(\kappa,u) = \frac{1}{2}\int_{u-\kappa}^{u+\kappa} du' \, \frac{\partial \phi}{\partial \kappa}(0,u') \, I_0\left(\sqrt{\kappa^2-(u-u')^2}\right) + \tag{124}$$

$$\frac{1}{2}\left[\phi(0,u-\kappa)+\phi(0,u+\kappa)\right] + \frac{1}{2}\int_{u-\kappa}^{u+\kappa} du' \, \phi(0,u') \, \frac{\kappa}{\sqrt{\kappa^2-(u-u')^2}} \, I_1\left(\sqrt{\kappa^2-(u-u')^2}\right),$$

where I_1 is the derivative of I_0. Its series expansion is:

$$I_1(z) = \sum_{j=0}^{\infty} \frac{(z/2)^{2j+1}}{j!\,(j+1)!}. \tag{125}$$

13.3. Reverse Transformation to the Original Equation

The initial conditions of the unknown ϕ have to be found from the initial conditions of $\rho(t,x)$ using the definitions (107), (110) and (111):

$$\rho(t,x) = \exp\left(-\frac{At}{2B}\right) \, \exp\left(-\frac{Cx}{2D}\right) \, \varphi(t,x) \qquad (107)$$

$$\kappa \equiv \alpha\, t, \;\; u \equiv \beta\, x, \;\; \phi(\kappa,u) \equiv \varphi(t(\kappa),x(u)), \qquad (110)$$

where

$$\alpha \equiv \sqrt{\frac{\frac{A^2}{4B}-\frac{C^2}{4D}-E}{B}}, \;\; \beta \equiv \sqrt{\frac{\frac{A^2}{4B}-\frac{C^2}{4D}-E}{D}}. \qquad (111)$$

If we set $t=0$ in (107):

$$\rho\left(0,\frac{u}{\beta}\right) = \exp\left(-\frac{Cu}{2D\beta}\right) \, \phi(0,u). \tag{126}$$

To find the initial condition $\frac{\partial \phi}{\partial \kappa}$ we need to take the time derivative of the definitions and set $t = 0$. This yields:

$$\frac{\partial \rho}{\partial t}\Big(0, \frac{u}{\beta}\Big) = \exp\left(-\frac{Cu}{2D\beta}\right)\ \left(\alpha\, \frac{\partial \phi}{\partial \kappa}(0,u) - \frac{A}{2B}\,\phi(0,u)\right). \qquad (127)$$

The two above equations constitute a system whose solutions are:

$$\phi(0,u) = \exp\left(\frac{Cu}{2D\beta}\right)\ \rho\Big(0, \frac{u}{\beta}\Big) = \exp\left(\frac{Cx}{2D}\right)\ \rho(0,x) \qquad (128)$$

and, likewise,

$$\frac{\partial \phi}{\partial \kappa}(0,u) = \frac{1}{\alpha}\ \exp\left(\frac{Cx}{2D}\right)\ \left[\frac{\partial \rho}{\partial t}(0,x) + \frac{A}{2B}\,\rho(0,x)\right]. \qquad (129)$$

In the solution (124) the terms $\phi(0, u \pm \kappa)$ appear. According to (128) they are:

$$\phi(0, u \pm \kappa) = \exp\left(\frac{Cu}{2D\beta}\right)\ \rho\Big(0, \frac{u \pm \kappa}{\beta}\Big) = \exp\left(\frac{Cx}{2D}\right)\ \rho\Big(0, x \pm \frac{\alpha}{\beta} t\Big). \qquad (130)$$

If we substitute the last 3 formulae in the solution (124) we obtain the formula for the general solution of the telegraph equation:

$$\rho(t,x) = \frac{1}{2}\ \exp\left(-\frac{A}{2B}\,t\right)\exp\left(-\frac{C}{2D}\,x\right)$$

$$\left[\frac{\beta}{\alpha}\int_{x-\frac{\alpha}{\beta}t}^{x+\frac{\alpha}{\beta}t} dx'\ \exp\left(\frac{Cx'}{2D}\right)\ \left[\frac{\partial \rho}{\partial t}(0,x') + \frac{A}{2B}\,\rho(0,x')\right]\ I_0\left(\sqrt{\alpha^2 t^2 - \beta^2 (x-x')^2}\right) + \right.$$

$$\left.\alpha\beta t\ \int_{x-\frac{\alpha}{\beta}t}^{x+\frac{\alpha}{\beta}t} \exp\left(\frac{Cx'}{2D}\right)\ \rho(0,x')\ \frac{I_1\left(\sqrt{\alpha^2 t^2 - \beta^2 (x-x')^2}\right)}{\sqrt{\alpha^2 t^2 - \beta^2 (x-x')^2}}\right] +$$

$$\frac{1}{2}\ \exp\left(-\frac{A}{2B}\,t\right)\left[\rho\Big(0, x - \frac{\alpha}{\beta}t\Big) + \rho\Big(0, x + \frac{\alpha}{\beta}t\Big)\right] \qquad (131)$$

13.4. Cases of Interest

In the telegrapher's equation that we are interested in, $A = \frac{1}{\tau}, B = 1, C = E = 0, D = v^2$. Then

$$\alpha = \sqrt{\frac{\frac{A^2}{4B} - \frac{C^2}{4D} - E}{B}} = \sqrt{\frac{1}{4\tau^2}} = \frac{1}{2\tau} \qquad (132)$$

and

$$\beta = \sqrt{\frac{\frac{A^2}{4B} - \frac{C^2}{4D} - E}{D}} = \sqrt{\frac{1}{4\tau^2 v^2}} = \frac{1}{2v\tau}. \tag{133}$$

The general solution becomes

$$\rho(t,x) = \frac{1}{2}\,\exp\left(-\frac{t}{2\tau}\right)$$

$$\left[\frac{1}{v}\int_{x-vt}^{x+vt} dx'\,\left[\frac{\partial\rho}{\partial t}(0,x') + \frac{1}{2\tau}\,\rho(0,x')\right]\, I_0\left(\frac{1}{2\tau}\sqrt{t^2 - \left(\frac{x-x'}{v}\right)^2}\right) + \right.$$

$$\left.\left(\frac{1}{2\tau}\right)^2 \frac{t}{v}\,\int_{x-vt}^{x+vt} dx'\,\rho(0,x')\,\frac{I_1\left(\frac{1}{2\tau}\sqrt{t^2-\left(\frac{x-x'}{v}\right)^2}\right)}{\frac{1}{2\tau}\sqrt{t^2-\left(\frac{x-x'}{v}\right)^2}}\right] +$$

$$\frac{1}{2}\,\exp\left(-\frac{t}{2\tau}\right)\left[\rho\Big(0, x-\frac{t}{v}\Big) + \rho\Big(0, x+\frac{t}{v}\Big)\right] \tag{134}$$

Substitution of the appropriate initial conditions yields the solutions (76) and (79).

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