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Chapter 14

RANDOM WALK IN A FINITE DIRECTED GRAPH SUBJECT TO A SYNCHRONIZING ROAD COLORING

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Abstract

A constructive proof is given to the fact that any ergodic Markov chain can be realized as a random walk subject to a synchronizing road coloring. Redundancy (ratio of extra entropy) in such a realization is also studied.

Keywords: Markov chain, random walk, entropy, road coloring, coupling from the past.

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1. Introduction

A random walk in $\mathbb{R}$ is a process $(S_n)_{n\geq 0}$ which may be represented as

$$S_n = \xi_n + \xi_{n-1} + \cdots + \xi_1 + S_0, \quad n \geq 1 \tag{1.1}$$

for some sequence $(\xi_n)_{n\geq 1}$ of IID (i.e., independent and identically distributed) random variables being independent of S_0. Note that equation (1.1) is equivalent to the recursion relation

$$S_n = \xi_n + S_{n-1}, \quad n \geq 1. \tag{1.2}$$

We may introduce a natural analogue of random walk taking values in a finite set V, say, $\{1,\ldots,m\}$. Let Σ denote the set of all mappings of V into itself. A *random walk in*

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V is a pair of processes $\{(X_n)_{n\ge 0}, (\phi_n)_{n\ge 1}\}$ such that $(\phi_n)_{n\ge 1}$ is a sequence of IID random variables taking values in Σ and being independent of X_0 and such that

$$X_n = (\phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_1)(X_0), \quad n \ge 1. \tag{1.3}$$

Note that equation (1.3) is equivalent to the recursion relation

$$X_n = \phi_n(X_{n-1}), \quad n \ge 1. \tag{1.4}$$

It is obvious that, for each $n \ge 1$, the random variable ϕ_n is independent of $\sigma(X_j, \phi_j : j \le n-1)$, since each X_j is measurable with respect to $\sigma(X_0, \phi_1, \ldots, \phi_j)$.

It is now natural to extend the index set to $\mathbb{Z}$, the set of all integers, as follows.

Definition 1.1. A *random walk in V parametrized by $\mathbb{Z}$* is a pair of processes $\{(X_n)_{n\in\mathbb{Z}}, (\phi_n)_{n\in\mathbb{Z}}\}$ which satisfies the following conditions:

(i) $(\phi_n)_{n\in\mathbb{Z}}$ is a sequence of IID random variables taking values in Σ;

(ii) for each $n \in \mathbb{Z}$, the random variable ϕ_n is independent of $\sigma(X_j, \phi_j : j \le n-1)$;

(iii) $X_n = \phi_n(X_{n-1})$ holds almost surely for all $n \in \mathbb{Z}$.

If ϕ_n's have common law μ on Σ, such a random walk is called a *μ-random walk*.

Our μ-random walk may also be called a *random walk in a finite directed graph subject to a road coloring*. The reason will be explained in Section 5. Each element of V will be called a *site*.

If a μ-random walk $\{(X_n)_{n\in\mathbb{Z}}, (\phi_n)_{n\in\mathbb{Z}}\}$ is given, then the process $(X_n)_{n\in\mathbb{Z}}$ is a Markov chain in V whose one-step transition probability is given by

$$P(X_1 = y | X_0 = x) = \mu(\sigma \in \Sigma : \sigma(x) = y), \quad x, y \in V. \tag{1.5}$$

Conversely, if a Markov chain $(Y_n)_{n\in\mathbb{Z}}$ is given, we call μ a *mapping law* for the Markov chain if the following identity holds:

$$P(Y_1 = y | Y_0 = x) = \mu(\sigma \in \Sigma : \sigma(x) = y), \quad x, y \in V. \tag{1.6}$$

In this case, for any μ-random walk $\{(X_n)_{n\in\mathbb{Z}}, (\phi_n)_{n\in\mathbb{Z}}\}$, we can easily show that the Markov chain $(X_n)_{n\in\mathbb{Z}}$ is identical in law to $(Y_n)_{n\in\mathbb{Z}}$.

Proposition 1.2. *For any Markov chain $(Y_n)_{n\in\mathbb{Z}}$ in V, there exists a mapping law μ.*

A proof by means of rational approximation can be found in [9]. We shall give its constructive proof in the next section.

Note that, for a given Markov chain, there may exist several mapping laws. We may expect that we can take a nice mapping law in the following sense.

Definition 1.3. Let μ be a probability law on Σ and denote by $\mathrm{Supp}(\mu)$ the support of μ. We say that μ is *synchronizing* (or simply *sync*) if there exists a finite sequence $\sigma_1, \ldots, \sigma_p$ of elements of $\mathrm{Supp}(\mu)$ such that $\sigma_p \circ \sigma_{p-1} \circ \cdots \circ \sigma_1$ maps V into a signleton.

Note that a μ-random walk associated with a sync mapping law is utilized in Propp–Wilson's sampling method of stationary law, which is called *coupling from the past*; we shall mention it briefly in Section 4.

Suppose that μ is sync and let $\{(X_n)_{n\in\mathbb{Z}}, (\phi_n)_{n\in\mathbb{Z}}\}$ be a μ-random walk. We may assume without loss of generality that for any $x \in V$ there exists $\sigma \in \mathrm{Supp}(\mu)$ such that $x \in \sigma(V)$; in fact, the Markov chain $(X_n)_{n\in\mathbb{Z}}$ never visits such sites $x \in V$ that $x \notin \sigma(V)$ for any $\sigma \in \mathrm{Supp}(\mu)$. Then we see that the Markov chain $(X_n)_{n\in\mathbb{Z}}$ is *ergodic*, i.e., the following two conditions hold (see, e.g., [5]):

(i) the Markov chain is *irreducible*, i.e., $P(X_0 = x) > 0$ for all $x \in V$ and for any $x, y \in V$ there exists $n \geq 1$ such that $P(X_n = y | X_0 = x) > 0$;

(ii) the Markov chain is *aperiodic*, i.e., for any $x \in V$, the greatest common divisor of $\{n \geq 1 : P(X_n = x | X_0 = x) > 0\}$ is one.

The condition (i) is obvious. The condition (ii) may be verified as follows. Let $x \in V$. Take $a \in V$ such that $\sigma_p \circ \cdots \circ \sigma_1(V) = \{a\}$ and take $q \geq 1$ such that $P(X_q = x | X_0 = a) > 0$. Then the set $\{n \geq 1 : P(X_n = x | X_0 = x) > 0\}$ contains all integers greater than $p + q$, and hence its greatest common divisor is one.

The following theorem asserts that the converse is also true.

Theorem 1.4 ([9]). *Suppose that $(Y_n)_{n\in\mathbb{Z}}$ is an ergodic Markov chain. Then there exists a sync mapping law.*

To prove Theorem 1.4, the authors in [9] utilized a profound graph-theoretic theorem, which was recently obtained by Trahtman [7], the complete solution to the *road coloring problem*; we shall explain it briefly in Section 5. In this chapter, we would like to give an elementary, self-contained and constructive proof of Theorem 1.4 without using Trahtman's theorem.

The remainder of this chapter is as follows. In Section 2, we give constructive proofs to Proposition 1.2 and Theorem 1.4. In Section 3, we study redundancy in random walk realization of a Markov chain. In Section 4, we mention the couplinng from the past. In Section 5, we explain how our random walk is related to road coloring. In Section 6, we provide a summary and conclusion.

2. A Constructive Proof of Existence of Sync Mapping Law

A matrix $Q = (q_{x,y})_{x,y\in V}$ with non-negative entries is called a *transition matrix* if

$$\sum_{y\in V} q_{x,y} = 1 \quad \text{for all } x \in V. \tag{2.7}$$

We give a constructive proof of Proposition 1.2 for later use.

A constructive proof of Proposition 1.2. It suffices to show that, for any transition matrix Q, there exists a mapping law μ for Q, i.e.,

$$q_{x,y} = \mu(\sigma \in \Sigma : \sigma(x) = y), \quad x, y \in V. \tag{2.8}$$

We define

$$E(Q) = \{(x,y) \in V \times V : q_{x,y} > 0\}. \tag{2.9}$$

Let us prove the result by induction of $\sharp E(Q)$, where $\sharp A$ stands for the number of elements of A. It is obvious by (2.7) that $\sharp\{y \in V : q_{x,y} > 0\} \geq 1$ for all $x \in V$, and hence that $\sharp E(Q) \geq \sharp V$.

Suppose that $\sharp E(Q) = \sharp V$. Then, by (2.7), it holds that $\sharp\{y \in V : q_{x,y} > 0\} = 1$ for all $x \in V$. This shows that there exists $\sigma \in \Sigma$ such that

$$q_{x,y} = \begin{cases} 1 & \text{if } y = \sigma(x), \\ 0 & \text{otherwise}. \end{cases} \tag{2.10}$$

Thus the Dirac mass at σ is as desired.

Let $k > \sharp V$ and suppose that all transition matrix Q such that $\sharp E(Q) < k$ admits a mapping law. Let Q be a transition matrix such that $\sharp E(Q) = k$. Write

$$\varepsilon = \min\{q_{x,y} : (x,y) \in E(Q)\}. \tag{2.11}$$

Since $\sharp E(Q) > \sharp V$, we see that $0 < \varepsilon < 1$. Take $(x,y) \in E(Q)$ such that $\varepsilon = q_{x,y}$, and take $\sigma \in \Sigma$ such that $(x,\sigma(x)) \in E(Q)$ for all $x \in V$. Define $\widetilde{Q} = (\widetilde{q}_{x,y})_{x,y \in V}$ by

$$\widetilde{q}_{x,y} = \frac{1}{1-\varepsilon}\left(q_{x,y} - \varepsilon 1_{\{\sigma(x)=y\}}\right). \tag{2.12}$$

Then we see that $\widetilde{Q}$ is a transition matrix and that $\sharp E(\widetilde{Q}) < k$. Now, by the assumption of the induction, we see that $\widetilde{Q}$ admits a mapping law $\widetilde{\mu}$. Therefore we conclude that $(1-\varepsilon)\widetilde{\mu} + \varepsilon\delta_\sigma$ is a mapping law for Q. The proof is now complete. □

Utilizing Proposition 1.2, we give a constructive proof of Theorem 1.4.

A constructive proof of Theorem 1.4. Let Q be the transition matrix of an ergodic Markov chain. Then we see that there exists $r \geq 1$ such that the r-th product Q^r has positive entries.

Take $x_0 \in V$ arbitrarily and set $V_r = \{x_0\}$. If V_k is defined for $k = r, r-1, \ldots, 1$, define V_{k-1} recursively by

$$V_{k-1} = \{x \in V : (x,y) \in E(Q) \text{ for some } y \in V_k\}, \tag{2.13}$$

where $E(Q)$ has been defined in (2.9). Note that

$$\sharp V_r \leq \sharp V_{r-1} \leq \cdots \leq \sharp V_0. \tag{2.14}$$

Since Q^r has positive entries, we see that $V_0 = V$.

For $k = r, r-1, \ldots, 1$, we pick $\sigma_k \in \Sigma$ so that $\sigma_k(x) \in V_k$ if $x \in V_{k-1}$ and $(x,\sigma_k(x)) \in E(Q)$ if $x \notin V_{k-1}$. We then have

$$\sigma_r \circ \sigma_{r-1} \circ \cdots \circ \sigma_1(V) = \{x_0\}. \tag{2.15}$$

Let $\mu^{(1)}$ denote the uniform law on $\{\sigma_1, \ldots, \sigma_r\}$. Then we see that $\mu^{(1)}$ is a sync mapping law for a transition matrix $Q^{(1)} = (q^{(1)}_{x,y})_{x,y \in V}$ where

$$q^{(1)}_{x,y} = \mu^{(1)}(\sigma \in \Sigma : y = \sigma(x)) = \frac{1}{r} \sum_{k=1}^{r} 1_{\{y=\sigma_k(x)\}}. \tag{2.16}$$

Write

$$\varepsilon = \min\{q_{x,y} : (x,y) \in E(Q)\} > 0 \tag{2.17}$$

and define $Q^{(2)} = (q^{(2)}_{x,y})_{x,y \in V}$ by

$$q^{(2)}_{x,y} = \frac{1}{1-\varepsilon} \left(q_{x,y} - \varepsilon q^{(1)}_{x,y} \right). \tag{2.18}$$

Then $Q^{(2)}$ is a transition matrix, so that we may obtain a mapping law $\mu^{(2)}$ for $Q^{(2)}$ in the constructive way of the proof of Proposition 1.2 given above.

Now we define

$$\mu = \varepsilon \mu^{(1)} + (1-\varepsilon)\mu^{(2)}, \tag{2.19}$$

which we have proved that is a sync mapping law for Q. The proof is therefore complete. □

3. Redundancy in Random Walk Realization

The uncertainty associated with information source may be measured by entropy (see, e.g., [2]). A Markov chain $Y = (Y_n)_{n \in \mathbb{Z}}$ with transition matrix $Q = (q_{x,y})_{x,y \in V}$ and with stationary law λ has its entropy given by

$$h(Y) = -\sum_{x,y \in V} \lambda(x) q_{x,y} \log q_{x,y}, \tag{3.20}$$

where we adopt the binary logarithm $\log = \log_2$ for simplicity, and follow the usual convention: $0 \log 0 = 0$. For a probability law μ on Σ, an IID sequence $\phi = (\phi_n)_{n \in \mathbb{Z}}$ with common law μ has its entropy given by

$$h(\phi) = h(\mu) = -\sum_{\sigma \in \Sigma} \mu(\sigma) \log \mu(\sigma). \tag{3.21}$$

A μ-random walk $(X, \phi) = \{(X_n)_{n \in \mathbb{Z}}, (\phi_n)_{n \in \mathbb{Z}}\}$ with stationary law λ is a Markov chain whose transition matrix $\overline{Q} = \left(\overline{q}_{(x,\nu),(y,\sigma)}\right)_{(x,\nu),(y,\sigma) \in V \times \Sigma}$ and stationary law $\overline{\lambda}$ given by

$$\overline{q}_{(x,\nu),(y,\sigma)} = \mu(\sigma) 1_{\{y=\sigma(x)\}}, \quad \overline{\lambda}((x,\nu)) = \mu(\nu)\lambda(w \in V : x = \nu(w)). \tag{3.22}$$

Now its entropy $h(X,\phi)$ is computed as

$$h(X,\phi) = -\sum_{(x,\nu),(y,\sigma)\in V\times\Sigma} \overline{\lambda}((x,\nu))\overline{q}_{(x,\nu),(y,\sigma)} \log \overline{q}_{(x,\nu),(y,\sigma)} \tag{3.23}$$

$$= -\sum_{x,y\in V,\ \sigma\in\Sigma} \left\{ \sum_{\nu\in\Sigma} \mu(\nu)\lambda(w\in V : x=\nu(w)) \right\} 1_{\{y=\sigma(x)\}}\mu(\sigma)\log\mu(\sigma) \tag{3.24}$$

$$= -\sum_{x\in V,\ \sigma\in\Sigma} \lambda(x) \left\{ \sum_{y\in V} 1_{\{y=\sigma(x)\}} \right\} \mu(\sigma)\log\mu(\sigma) \tag{3.25}$$

$$= -\left\{ \sum_{x\in V} \lambda(x) \right\} \left\{ \sum_{\sigma\in\Sigma} \mu(\sigma)\log\mu(\sigma) \right\} \tag{3.26}$$

$$= -\sum_{\sigma\in\Sigma} \mu(\sigma)\log\mu(\sigma). \tag{3.27}$$

Thus we obtain $h(X,\phi) = h(\phi) = h(\mu)$.

If the Markov chain Y is identical in law to X for some μ-random walk (X,ϕ), we have

$$h(\mu) \geq h(Y). \tag{3.28}$$

In fact, by (2.8), we have $\mu(\sigma) \leq q_{x,y}$ if $y = \sigma(x)$, and hence we see that

$$h(\mu) = -\sum_{x\in V} \lambda(x) \sum_{\sigma\in\Sigma} \mu(\sigma)\log\mu(\sigma) \tag{3.29}$$

$$= -\sum_{x,y\in V} \lambda(x) \sum_{\sigma\in\Sigma: y=\sigma(x)} \mu(\sigma)\log\mu(\sigma) \tag{3.30}$$

$$\geq -\sum_{x,y\in V} \lambda(x) \sum_{\sigma\in\Sigma: y=\sigma(x)} \mu(\sigma)\log q_{x,y} \tag{3.31}$$

$$= -\sum_{x,y\in V} \lambda(x) q_{x,y}\log q_{x,y} = h(Y). \tag{3.32}$$

The inequality (3.28) shows that any μ-random walk realization of Y requires some extra entropy, the extent of which may be measured by

$$r(\mu;Y) := \frac{h(\mu)-h(Y)}{h(\mu)}. \tag{3.33}$$

This quantity $r(\mu;Y)$ is called the *(relative) redundancy* in the μ-random walk realization of the Markov chain Y. We denote the totality of all possible redundancies by

$$\rho(Y) = \{r(\mu;Y) : \ \mu \text{ is a mapping law for } Y\}. \tag{3.34}$$

Theorem 3.1. *For a Markov chain Y, the following assertions hold:*

(i) the set $\rho(Y)$ has finite minimum $r(Y) \geq 0$ and maximum $R(Y) \leq 1$;

(ii) for any $r(Y) \leq r \leq R(Y)$, there exists a mapping law μ for Y such that $r(\mu;Y) = r$.

Moreover, if Y is ergodic, then the following assertion also holds:

(iii) for any $r(Y) < r < R(Y)$*, there exists a sync mapping law* μ *for* Y *such that* $r(\mu;Y) = r$*.*

Proof. Let us remark on several basic facts about the entropy. Since Σ is a finite set, the totality of probability measures on Σ, which is denoted by $\mathcal{P}(\Sigma)$, is equipped with the total variation topology. It is well-known that $\mathcal{P}(\Sigma)$ is compact and that $\mu_n \to \mu$ if and only if $\mu_n(\sigma) \to \mu(\sigma)$ for all $\sigma \in \Sigma$. By definition (3.21), the function $\mathcal{P}(\Sigma) \ni \mu \mapsto h(\mu)$ is continuous.

(i) Let $\mathcal{P}(Y)$ denote the set of all mapping laws for Y. It is obvious that $\mathcal{P}(Y)$ is a compact convex subset of $\mathcal{P}(\Sigma)$. Since $h(\mu) \geq h(Y) > 0$ for all $\mu \in \mathcal{P}(Y)$, and since $t \mapsto (t - h(Y))/t$ is continuous in $t \geq h(Y)$, we see that $\mathcal{P}(Y) \ni \mu \mapsto r(\mu;Y)$ is continuous. Hence we see that the set $\rho(Y)$ has finite minimum $r(Y)$ and maximum $R(Y)$.

(ii) Take $\mu^{(1)}, \mu^{(2)} \in \mathcal{P}(Y)$ such that $r(Y) = r(\mu^{(1)};Y)$ and $R(Y) = r(\mu^{(2)};Y)$. Let $0 \leq p \leq 1$. Then $\mu_p := p\mu^{(1)} + (1-p)\mu^{(2)}$ also belongs to $\mathcal{P}(Y)$. Since $[0,1] \ni p \mapsto r(\mu_p;Y)$ is continuous, we see that $\rho(Y)$ contains all r such that $r(Y) < r < R(Y)$. Thus we obtain (ii).

(iii) Suppose that Y is ergodic. Theorem 1.4 implies that there exists a sync mapping law $\mu^{(0)}$ for Y. Let $r(Y) < r < R(Y)$ and take $r^{(1)}$, $r^{(2)}$ such that $r(Y) < r^{(1)} < r$ and $r < r^{(2)} < R(Y)$. By (ii), we may take mapping laws $\mu^{(1)}$ and $\mu^{(2)}$ for Y such that $r(\mu^{(1)};Y) = r^{(1)}$ and $r(\mu^{(2)};Y) = r^{(2)}$. Now we may take $\varepsilon > 0$ small enough such that

$$r((1-\varepsilon)\mu^{(1)} + \varepsilon\mu^{(0)};Y) < r < r((1-\varepsilon)\mu^{(2)} + \varepsilon\mu^{(0)};Y). \tag{3.35}$$

Hence we may take $0 < p < 1$ such that the mapping law μ defined by

$$\mu = (1-\varepsilon)(p\mu^{(1)} + (1-p)\mu^{(2)}) + \varepsilon\mu^{(0)} \tag{3.36}$$

satisfies $r(\mu;Y) = r$. This shows that μ is a sync mapping law for Y. Therefore the proof is complete. □

Example 3.2. Let $V = \{1,2,3\}$ and let

$$Q = \begin{bmatrix} q_{1,1} & q_{1,2} & q_{1,3} \\ q_{2,1} & q_{2,2} & q_{2,3} \\ q_{3,1} & q_{3,2} & q_{3,3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}. \tag{3.37}$$

The Markov chain Y with transition matrix Q has a unique stationary law

$$\lambda = \left[\lambda(1), \lambda(2), \lambda(3)\right] = \frac{1}{9}\left[3,2,4\right]. \tag{3.38}$$

A simple computation leads to $h(Y) = 2/3$. For a mapping law μ for Y, elements which may possibly be contained in $\mathrm{Supp}(\mu)$ are the following four:

$$\sigma^{(1)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \sigma^{(2)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \sigma^{(3)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \sigma^{(4)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{3.39}$$

Set $p = \mu(\sigma^{(1)})$. A simple computation leads to

$$\mu(\sigma^{(2)}) = p, \quad \mu(\sigma^{(3)}) = \mu(\sigma^{(4)}) = 1/2 - p. \tag{3.40}$$

Thus we obtain

$$h(\mu) = 2f(p) + 2f(1/2 - p), \tag{3.41}$$

where $f(t) = -t\log t$. Since the variable p may vary in $[0, 1/2]$, we see that $h(\mu)$ ranges $[1, 2]$, where the minimum $h(\mu) = 1$ is attained at $p = 0$ and $1/2$ and the maximum $h(\mu) = 2$ at $p = 1/4$. Hence we obtain

$$r(Y) = 1/3, \quad R(Y) = 2/3. \tag{3.42}$$

In this case, for all $p \in [0, 1/2]$, the mapping law μ is sync; in fact, $\sigma^{(1)} \circ \sigma^{(2)}(V) = \{1\}$ and $\sigma^{(3)} \circ \sigma^{(4)} \circ \sigma^{(4)} \circ \sigma^{(3)}(V) = \{3\}$.

Before closing this section, we mention the following theorem, which provides a necessary and sufficient condition for zero minimum redundancy.

Theorem 3.3 ([9]). *Suppose that Y is ergodic. Then $r(Y) = 0$ if and only if Y is* p-uniform, *i.e., there exists a probability law ν on V and a family $\{\tau_x : x \in V\}$ of permutations of V such that*

$$q_{x,y} = \nu(\tau_x(y)), \quad x, y \in V. \tag{3.43}$$

For the proof of Theorem 3.3, see [9].

4. Coupling from the Past

In some practical problems, we sometimes need to simulate the stationary law of an ergodic Markov chain. As a powerful method for the simulation, Propp–Wilson's coupling from the past is widely known; see [6] and also [3] and [4]. The fundamental idea is to utilize a random walk realization associated with a sync mapping law. Let us explain it briefly.

Let an ergodic Markov chain be given and suppose that we find a sync mapping law μ for the Markov chain. Then a μ-random walk $\{(X_n)_{n\in\mathbb{Z}}, (\phi_n)_{n\in\mathbb{Z}}\}$ is a realization of the Markov chain. Let $(\sigma_1, \ldots, \sigma_p)$ be a finite sequence of elements of $\mathrm{Supp}(\mu)$ such that $\sigma_p \circ \cdots \circ \sigma_1(V)$ is a singleton. The latest time when the exact sequence $(\sigma_p, \sigma_{p-1}, \ldots, \sigma_1)$ can be found in $(\phi_0, \phi_{-1}, \ldots)$ will be denoted by

$$T = \sup\{k \in \mathbb{Z} : 0 \geq k + p - 1,\ \phi_{k+p-1} = \sigma_p, \ldots, \phi_k = \sigma_1\}. \tag{4.44}$$

Here we understand that $\sup \emptyset = -\infty$. Note that T is finite almost surely. This random time T plays a role of stopping time in the sense that

$$\{T = k\} \in \sigma(\phi_0, \phi_{-1}, \ldots, \phi_k) \quad \text{for } 0 \geq k + p - 1. \tag{4.45}$$

Since $\sigma_p \circ \cdots \circ \sigma_1(V)$ is a singleton, we see that $\phi_0 \circ \phi_{-1} \circ \cdots \circ \phi_T$ maps V into a singleton. Thus it holds that

$$X_0 = \phi_0 \circ \phi_{-1} \circ \cdots \circ \phi_T(x) \quad \text{a.s.} \tag{4.46}$$

for all $x \in V$. This shows the folloiwng: We pick a sequence $f_0, f_{-1}, \ldots$ from the law μ up to the latest time T when $(f_{T+p-1}, \ldots, f_T) = (\sigma_p, \ldots, \sigma_1)$. Then the resulting site $f_0 \circ f_{-1} \circ \cdots \circ f_T(x)$, which does not depend on the choice of $x \in V$, is a sample point from the stationary law, which is as desired.

This method can be applied to simulate a Gibbs distribution. In this case, a sync mapping law can be constructed with the help of monotonicity structure of the state space V.

Remark 4.1. The identity (4.46) implies that, for each $n \in \mathbb{Z}$, the random variable X_n is measurable with respect to $\sigma(\phi_j : j \le n)$. One can ask what happens when μ is not sync. The following theorem answers this question.

Theorem 4.2 (Yano [8]). *Suppose that the Markov chain $(X_n)_{n\in\mathbb{Z}}$ is ergodic. and that μ is not sync. Then, for each $n \in \mathbb{Z}$, the random variable X_n is not measurable with respect to $\sigma(\phi_j : j \le n)$.*

For the proof of Theorem 4.2, see [8].

5. Random Walk and Road Coloring

Let us explain how our μ-random walk is related to road coloring.

First, let us introduce some notations in graph theory. A finite directed graph is the pair (V,A) of finite sets V and A associated with mappings $i : A \to V$ and $t : A \to V$. Each element of V will be called a *site* (or a *node*) and each element a of A will be called a *(oneway) road* (or an *arrow*) which runs from $i(a)$ to $t(a)$. For $a \in A$, the site $i(a)$ (resp. $t(a)$) will be called the *initial* (resp. *terminal*) site of a. For $x \in V$, the number of roads running from x, namely,

$$O(x) = \sharp\{a \in A : i(a) = x\}, \tag{5.47}$$

will be called the *outdegree* at the site x. If $O(x)$ does not depend on $x \in V$, the directed graph (V,A) is called *of constant outdegree*. A *path* from $x \in V$ to $y \in V$ is a word $w = (a_1, \ldots, a_n)$ of roads such that a_1 runs from x to $i(a_2)$, a_2 to $i(a_3)$, …, a_{n-1} to $i(a_n)$, and a_n to y. The number $L(w) = n$ is called the *length* of the path $w = (a_1, \ldots, a_n)$. The directed graph (V,A) is called *strongly connected* if, for any $x, y \in V$, there exists a path from x to y. The directed graph (V,A) is called *aperiodic* if, for any $x \in V$, the greatest common divisor of the set of $L(w) \ge 1$ among all paths w from x to itself is one.

Second, we introduce some notations in road coloring. Suppose that (V,A) is of constant outdegree and denote the common outdegree by d. A *road coloring* of (V,A) is a partition of A into d disjoint subsets $C = \{c^{(1)}, \ldots, c^{(d)}\}$ such that, for each $x \in V$, each *color* $c^{(k)}$ contains one and only one road whose initial site is x. For a finite sequence $s = (c_1, \ldots, c_p)$ of elements of C, a path $w = (a_1, \ldots, a_p)$ is said to be *along* s if $a_k \in c_k$ for all $k = 0, 1, \ldots, p$. The following notion originates Adler, Goodwyn and Weiss [1].

Definition 5.1. A road coloring C of (V,A) is called *sync* if there exists a finite sequence $c_1, \ldots, c_p$ of elements of C such that all paths along $(c_1, \ldots, c_p)$ have common terminal site.

Let us give an example.

Example 5.2. Let $V = \{1,2,3\}$ and $A = \{a^{(x,k)} : x \in V,\ k = 1,2\}$ and define the initial and terminal sites of each road as follows:

a	$a^{(1,1)}$	$a^{(2,1)}$	$a^{(3,1)}$	$a^{(1,2)}$	$a^{(2,2)}$	$a^{(3,2)}$
$i(a)$	1	2	3	1	2	3
$t(a)$	3	3	1	3	1	2

(5.48)

Take the road coloring $C = \{c^{(1)}, c^{(2)}\}$ defined by

$$c^{(1)} = \{a^{(x,1)} : x \in V\}, \quad c^{(2)} = \{a^{(x,2)} : x \in V\}. \tag{5.49}$$

Now it is obvious that the road coloring C is sync; in fact, all paths along $(c^{(1)}, c^{(2)}, c^{(2)}, c^{(1)})$ have common terminal site 3.

Third, we recall the *road coloring problem*. If a directed graph (V,A) of constant outdegree admits a sync road coloring, then it is necessarily strongly connected and aperiodic. The converse was posed as a conjecture by Adler, Goodwyn and Weiss [1], which had been called the road coloring problem until it was completely solved by Trahtman [7].

Theorem 5.3 (Trahtman [7]). *A directed graph which is of constant outdegree, strongly connected, and aperiodic, does admit a sync road coloring.*

Fourth, let us explain how to understand our μ-random walk by means of road coloring. Let μ be a probability law on Σ. Since Σ is a finite set, the support of μ may be written as $\{\sigma^{(1)}, \dots, \sigma^{(d)}\}$. We define the set A of roads as the totality of $a^{(x,k)}$ for $x \in V$ and $k = 1, \dots, d$ where $a^{(x,k)}$ runs from x to $\sigma^{(k)}(x)$. Thus the law μ induces naturally the road coloring $C = \{c^{(1)}, \dots, c^{(d)}\}$ such that

$$c^{(k)} = \{a^{(x,k)} : x \in V\}. \tag{5.50}$$

It is now obvious that the probability law μ is sync in the sense of Definition 1.3 if and only if the road coloring C is sync in the sense of Definition 5.1.

For a μ-random walk (X, ϕ), the process X moves from site to site in the directed graph (V,A) via the equation $X_n = \phi_n(X_{n-1})$, being driven by the colors of roads indicated by ϕ which are randomly chosen from the road coloring C induced by μ. Thus we may call (X, ϕ) a μ-random walk in the directed graph (V,A) subject to the road coloring C.

Let Y be a Markov chain and suppose that Y is realized as X of a μ-random walk (X, ϕ) in the directed graph (X, ϕ) subject to the road coloring C induced by μ. Then, to each edge $(x,y) \in E(Y)$, there corresponds at least one road a which runs from x to y. For example, consider Example 3.2 with $p = 0$. In this case, we have $\mathrm{Supp}(\mu) = \{\sigma^{(3)}, \sigma^{(4)}\}$, and

$$E(Y) = \{(1,3),(2,1),(2,3),(3,1),(3,2)\}. \tag{5.51}$$

The set A of roads induced by μ is given as $A = \{a^{(x,k)} : x \in V, k = 1,2\}$, where the initial and terminal sites of each road are given as (5.48). Then we find that the road coloring induced by μ is nothing else but $C = \{c^{(1)}, c^{(2)}\}$ given as (5.49) in Example 5.2, where we note that $\sigma^{(3)}$ and $\sigma^{(4)}$ correspond to $c^{(1)}$ and $c^{(2)}$, respectively. Note that there exist two roads $a^{(1,1)}$ and $a^{(1,2)}$ which run from 1 to 3, which are colored differently from each other. See Figure 1 below for the illustration.

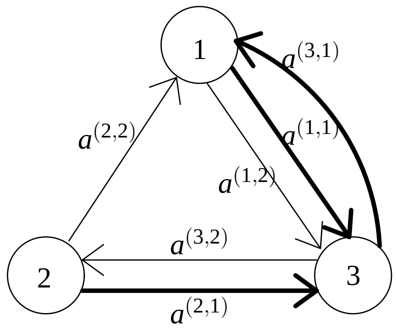

Figure 1.

6. Conclusion

We have introduced a random walk in a finite set as a stochastic evolutionary process driven by an IID sequence of mappings. It can be understood as a random walk in a finite directed graph moving according to random road colors. Any ergodic Markov chain is proved to be realized, in a constructive way, by a random walk associated with a sync mapping law. The redundancy in random walk realization with a sync mapping law can be as close as desired to the minimum redundancy.

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