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Chapter 4

TAIL VALUE AT RISK. AN ANALYSIS WITH THE NORMAL-POWER APPROXIMATION

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Abstract

The problem of risk measurement is one of the most important problems in the risk management. In this chapter we discuss risk measures based on loss distributions in the context of insurance and finance. We concentrate on the most known risk measures: the Value at Risk and the Tail Value at Risk. The aim of this chapter is to analyze the applicability of the Normal-Power approximation for the calculus of TVaR. We obtain a new analytical expression of the TVaR using the NP approximation and we analyze its precision. The chapter ends up with an application to underwriting and credit risk.

Keywords: Tail Value at Risk, Value at Risk, Normal-Power approximation

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1. Introduction

The problem of risk measurement is one of the most important problems in the risk management. In this chapter we discuss risk measures based on loss distributions in the context of insurance and finance. We concentrate on the most known risk measures: the Value at Risk and the Tail Value at Risk. A detailed analysis of these and other risk measures including their properties can be found in a seminal paper [1] and several books (e.g. [7] and [9]).

The Normal-Power (NP) approximation has been used in the insurance field from [8] to approach the distribution of aggregate claims. A very good and extensive explanation of the background, derivation an applicability of this NP approximation can be found in [3]. The NP approximation is usually compared with the Normal one, because the main difference between the two is that the NP includes the skewness of the risk. It has been early applied to approach the quantiles of aggregate claims, that is, in modern nomenclature, the Value at Risk.

In this chapter, we consider the application of the NP approximation to the calculus of the Tail Value at Risk. These two risk measures, Value at Risk (*VaR*) and Tail Value at Risk (*TVaR*), are nowadays very important from a practical point of view, because they are used in the new rules for controlling the solvency of financial and insurance institutions. From a solvency point of view, the *VaR* for a confidence level α is the value of the loss such that the probability that the loss is greater than this value, is at most $1-\alpha$. This measure does not give any information about the severity of losses which occur with a probability less than $1-\alpha$. With the *TVaR* this aspect is covered because, instead of fixing a concrete confidence level α, we average *VaR* over all levels greater than or equal to α. Figure 1 represents the concept of *VaR* and *TVaR* using the density function of the risk.

Lets us introduce the formal definitions of *VaR*, *TVaR* and two additional related risk measures.

Definition 1. *Let X be a random variable that represents a risk. The Value at Risk of X at a confidence level α, $VaR_X(\alpha)$, is*

$$VaR_X(\alpha) = \inf\{x : P(X \le x) \ge \alpha\}. \tag{1}$$

Definition 2. *Let X be a random variable that represents a risk. The Tail Value*

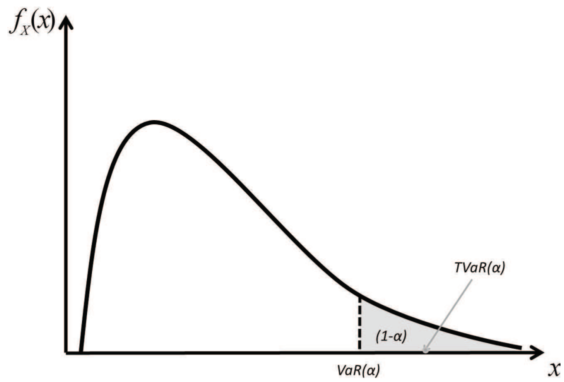

Figure 1. Graphical definition of *VaR* and *TVaR*.

at Risk of X at a confidence level α*,* $TVaR_X(\alpha)$*, is*

$$TVaR_X(\alpha) = \frac{1}{1-\alpha}\int_{\alpha}^{1} VaR_X(s)\,ds. \tag{2}$$

Definition 3. *Let X be a random variable that represents a risk. The Expected Shortfall of X at a confidence level* α*,* $ES_X(\alpha)$*, is*

$$ES_X(\alpha) = E\left[(X - VaR_X(\alpha))^+\right]. \tag{3}$$

Definition 4. *Let X be a random variable that represents a risk. The Conditional Tail Expectation of X at a confidence level* α*,* $CTE_X(\alpha)$*, is*

$$CTE_X(\alpha) = E\left[X \,|X > VaR_X(\alpha)\right]. \tag{4}$$

Other expressions for $TVaR_X(\alpha)$ and $CTE_X(\alpha)$ are

$$TVaR_X(\alpha) = VaR_X(\alpha) + \frac{1}{1-\alpha} ES_X(\alpha),$$

$$CTE_X(\alpha) = VaR_X(\alpha) + \frac{1}{1 - F_X(VaR_X(\alpha))} ES_X(\alpha).$$

From these two last relations, we observe that, if the distribution function of X is continuous in the value $VaR_X(\alpha)$, i.e. $F_X(VaR_X(\alpha)) = \alpha$, then $CTE_X(\alpha) =$

$TVaR_X(\alpha)$. Thus, for continuous random variables, the Tail Value at Risk and the Conditional Tail Expectation coincides.

The main aim of this chapter is to analyze the applicability of the NP approximation for the calculus of *TVaR*. In order to attain this objective, the chapter is structured as follows: after this introduction, in Section 2 we present a new analytical expression of the *TVaR* using the NP approximation; in Section 3 we analyze the precision of the approximation considering three distributions for the risk (exponential, Pareto and lognormal) and in Section 4 we include an application to the measurement of underwriting and credit risk. After some conclusions, an Annex with proofs is added.

2. Tail Value at Risk for the Normal-Power Approximation

In this section, we present a new analytical expression of the *TVaR* of a random variable when we use the Normal-Power approximation for its distribution. Before obtaining it, we need two lemmas, Lemma 1 for the *TVaR* of a $N(0,1)$ random variable (r.v.) and Lemma 2 for the *VaR* of a continuous r.v. with the NP approximation. These two results can be found also in several manuals (see e.g. [3] and [7]).

Lemma 1. *Let $Y \sim N(0,1)$, with distribution function $\Phi(y)$ and density function $\phi(y) = \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}}$. Let z_α be the Value at Risk of Y at a confidence level α, $VaR_Y(\alpha) = \Phi^{-1}(\alpha) = z_\alpha$. Its TVaR at a confidence level α is*

$$TVaR_Y(\alpha) = \frac{\phi(z_\alpha)}{1-\alpha}. \tag{5}$$

Proof.

$$\begin{aligned} TVaR_Y(\alpha) &= \frac{1}{1-\alpha}\int_\alpha^1 VaR_Y(s)ds = \frac{1}{1-\alpha}\int_{z_\alpha}^\infty y dF_Y(y) \\ &= \frac{1}{1-\alpha}\int_{z_\alpha}^\infty y \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} dy = \frac{1}{1-\alpha}\frac{1}{\sqrt{2\pi}} e^{\frac{-z_\alpha^2}{2}} \\ &= \frac{\phi(z_\alpha)}{1-\alpha}. \end{aligned}$$

□

Consider now a continuous r.v. X and that we use the NP approximation for its distribution. Then,

$$F_X(x) \simeq \Phi\left(\sqrt{\frac{9}{\gamma_1^2}+\frac{6}{\gamma_1}\left(\frac{x-E[X]}{\sqrt{V[X]}}\right)+1}-\frac{3}{\gamma_1}\right),$$

being $\gamma_1 = \frac{E\left[(X-E[X])^3\right]}{\left(\sqrt{V[X]}\right)^3}$ the skewness of X.

Lemma 2. *Let X be a continuous random variable. Its VaR at a confidence level α using the NP approximation is*

$$VaR_X(\alpha) \simeq E[X] + \sqrt{V[X]}\left(z_\alpha + \frac{\gamma_1}{6}\left(z_\alpha^2 - 1\right)\right). \tag{6}$$

Proof. Let X be a continuous random variable. Define $XN = \frac{X-E[X]}{\sqrt{V[X]}}$. From the definition of *VaR*,

$$VaR_X(\alpha) \simeq E[X] + \sqrt{V[X]}VaR_{XN}(\alpha).$$

If we use the NP approximation for X, the distribution function of XN is $F_{XN}(x) \simeq \Phi\left(\sqrt{\frac{9}{\gamma_1^2}+\frac{6}{\gamma_1}x+1}-\frac{3}{\gamma_1}\right)$, being γ_1 the skewness of X and also of XN. The *VaR* of XN is defined as

$$\begin{aligned} F_{XN}(VaR_{XN}(\alpha)) &= P[XN \leq VaR_{XN}(\alpha)] \\ &\simeq \Phi\left(\sqrt{\frac{9}{\gamma_1^2}+\frac{6}{\gamma_1}VaR_{XN}(\alpha)+1}-\frac{3}{\gamma_1}\right) = \alpha, \end{aligned}$$

then $\sqrt{\frac{9}{\gamma_1^2}+\frac{6}{\gamma_1}VaR_{XN}(\alpha)+1}-\frac{3}{\gamma_1} = z_\alpha$, and isolating $VaR_{XN}(\alpha)$ the Lemma is proved. □

Let us present now, in the following theorem, the main result of this chapter.

Theorem 1. *Let X be a continuous r.v. Its TVaR at a confidence level α using the NP approximation is*

$$TVaR_X(\alpha) \simeq E[X] + \sqrt{V[X]}\frac{\phi(z_\alpha)}{1-\alpha}\left(1+\frac{\gamma_1}{6}z_\alpha\right). \tag{7}$$

Proof. Let X be a continuous r.v. Define $XN = \frac{X-E[X]}{\sqrt{V[X]}}$. From the definition of *TVaR*,

$$TVaR_X(\alpha) \simeq E[X] + \sqrt{V[X]}\,TVaR_{XN}(\alpha).$$

If we use the NP approximation for X, the distribution function of XN is $F_{XN}(x) \simeq \Phi\left(\sqrt{\frac{9}{\gamma_1^2}+\frac{6}{\gamma_1}x+1}-\frac{3}{\gamma_1}\right)$, being γ_1 the skewness of X and also of XN. The *TVaR* of XN is $TVaR_{XN}(\alpha) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_{XN}(s)ds$. Considering Lemma 2,

$$\begin{aligned} TVaR_{XN}(\alpha) &\simeq \frac{1}{1-\alpha}\int_\alpha^1 \left(z_s + \frac{\gamma_1}{6}\left(z_s^2-1\right)\right)ds \\ &= \frac{1}{1-\alpha}\int_\alpha^1 z_s ds - \frac{1}{1-\alpha}\int_\alpha^1 \frac{\gamma_1}{6}ds + \frac{\gamma_1}{6(1-\alpha)}\int_\alpha^1 z_s^2 ds \\ &= \frac{\phi(z_\alpha)}{1-\alpha} - \frac{\gamma_1}{6} + \frac{\gamma_1}{6(1-\alpha)} I(\alpha), \end{aligned} \tag{8}$$

being

$$\begin{aligned} I(\alpha) &= \int_\alpha^1 z_s^2 ds = \{s = \Phi(y)\} = \int_{z_\alpha}^\infty y^2 \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} dy \\ &= \left[-y\frac{1}{\sqrt{2\pi}}e^{\frac{-y^2}{2}}\right]_{z_\alpha}^\infty + \int_{z_\alpha}^\infty \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} dy = \phi(z_\alpha) z_\alpha + (1-\alpha). \end{aligned} \tag{9}$$

Then, substituting (9) in (8) the proof is completed. □

3. Analysis of the Precision of the Approximation

In this section, in order to evaluate the precision of the expression obtained for the *TVaR* using the NP approximation, we will consider three usual distributions in the insurance and risk field: exponential, lognormal and Pareto. The common characteristic of these distributions is that we can find exact expressions for their *VaR* and *TVaR* (the proofs of these expressions are included in the Annex to this chapter). Then, we compare the exact expressions with the approached expressions given by the NP, for different confidence levels.

3.1. Exponential Distribution

Let X be an exponentially distributed r.v. with parameter β, $X \sim Exp(\beta)$. Then, $E[X] = \frac{1}{\beta}$, $V[X] = \frac{1}{\beta^2}$ and $\gamma_1 = 2$. The expressions of *VaR* and *TVaR* are

$$VaR_X(\alpha) = -\frac{\ln(1-\alpha)}{\beta}, \tag{10}$$

$$TVaR_X(\alpha) = -\frac{\ln(1-\alpha)}{\beta} + \frac{1}{\beta}. \tag{11}$$

The approximations given by the NP, using (6) and (7) are

$$VaR_X(\alpha) \simeq \frac{1}{\beta}\left(\frac{2}{3} + z_\alpha + \frac{z_\alpha^2}{3}\right), \tag{12}$$

$$TVaR_X(\alpha) \simeq \frac{1}{\beta}\left(1 + \frac{\phi(z_\alpha)}{1-\alpha}\left(1 + \frac{z_\alpha}{3}\right)\right). \tag{13}$$

In an insurance context, from a solvency point of view, we are interested in very high values of α. The values (exact and approached using NP) of $VaR_X(\alpha)$ and $TVaR_X(\alpha)$ for $\beta = 1$ and confidence levels $0.9, 0.95, 0.99, 0.995$ are included in Table 1. As the two risk measures are in this case proportional to the risk expectation, we compute only the standard case $\beta = 1$.

Table 1. Exponential distribution. Exact and approached values of *VaR* and *TVaR* for different confidence levels and $E[X] = 1$

$E[X]=1 \quad V[X]=1 \quad \gamma_1 = 2$				
	$Exp(1)$		Normal Power	
α	(10) $VaR_X(\alpha)$	(11) $TVaR_X(\alpha)$	(12) $VaR_X(\alpha)$	(13) $TVaR_X(\alpha)$
0.995	5.298	6.248	5.454	6.375
0.99	4.605	5.605	4.797	5.732
0.95	2.996	3.996	3.213	4.194
0.9	2.303	3.303	2.496	3.505

Let us now define the differences between the exact values and the approached ones using the NP, $DVaR_X(\alpha)$ and $DTVaR_X(\alpha)$,

$$DVaR_X(\alpha) = \frac{1}{\beta}\left(-\ln(1-\alpha) - \frac{2}{3} - z_\alpha - \frac{z_\alpha^2}{3}\right),$$

$$DTVaR_X(\alpha) = \frac{1}{\beta}\left(-\ln(1-\alpha) - \frac{\phi(z_\alpha)}{1-\alpha}\left(1+\frac{z_\alpha}{3}\right)\right).$$

These differences depend on the parameter of the exponential distribution, β, and the confidence level α; they are proportional to the expectation of the risk. In Figure 2 and 3 we represent $DVaR_X(\alpha)/E[X]$ and $DTVaR_X(\alpha)/E[X]$ as functions of the confidence level α.

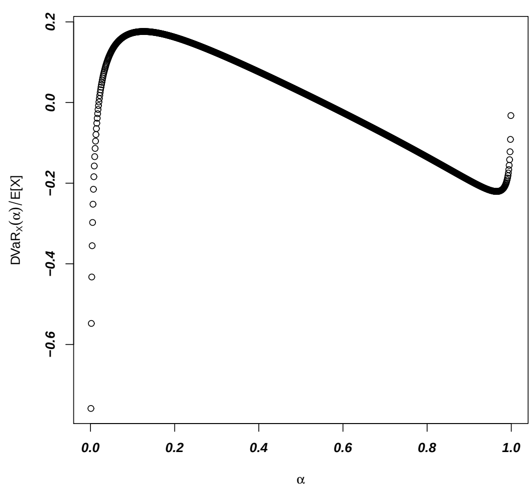

Figure 2. $\frac{DVaR_X(\alpha)}{E[X]}$ as a function of the confidence level α.

In Figure 2 we observe that there exist three values of the confidence level α that let $DVaR_X(\alpha^*)/E[X] = 0$, i.e, for these α^*, the NP approximation is equal to the exact value. The three roots α^* are $\alpha_1^* \simeq 0.0198$, $\alpha_2^* \simeq 0.5518$ and $\alpha_3^* \simeq 0.9993$. For $\alpha \in ((0,\alpha_1^*) \cup (\alpha_2^*, \alpha_3^*))$ the values of $DVaR_X(\alpha)/E[X]$ are negative, i.e. the NP approximation for *VaR* gives higher values than the exact ones. If we are interested in solvency problems, we look at high confidence levels, in this case, greater than α_2^*. Then, if $0.5518 < \alpha < 0.9993$, NP approximation overestimates the *VaR* and a local minimum of $DVaR_X(\alpha)/E[X]$ is attained at $\alpha = 0.965$. If $\alpha > 0.9993$, NP approximation underestimates the *VaR*.

As in the previous case, there are three values of the confidence level that

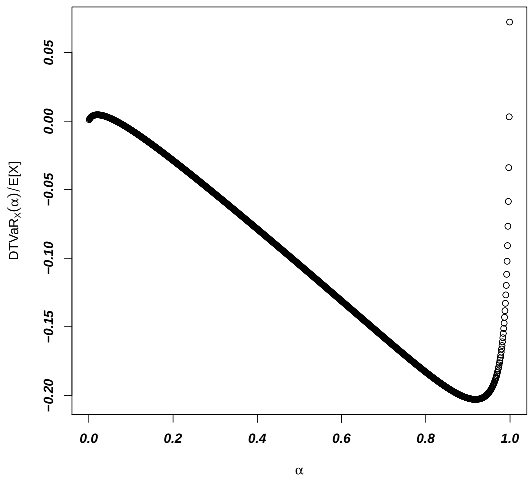

Figure 3. $\frac{DTVaR_X(\alpha)}{E[X]}$ as a function of the confidence level α.

let $DTVaR_X(\alpha^*)/E[X] = 0$. From the graph we see that NP approximation overestimates the VaR for high confidence levels but lower than 0.9178 and underestimates it for confidence levels higher than this value.

3.2. Pareto Distribution

Let X be a Pareto distributed r.v., $X \sim Pa(a,s)$. Then,

$$E[X] = \frac{s}{a-1} \quad (a > 1),$$

$$V[X] = \frac{s^2 a}{(a-1)^2 (a-2)} \quad (a > 2) \text{ and}$$

$$\gamma_1 = \frac{2\sqrt{a-2}\,(a+1)}{(a-3)\sqrt{a}} \quad (a > 3).$$

The expressions of VaR and $TVaR$ are

$$VaR_X(\alpha) = \frac{s}{(1-\alpha)^{\frac{1}{a}}} - s, \tag{14}$$

$$TVaR_X(\alpha) = \frac{s}{a-1} + \frac{a}{a-1} VaR_X(\alpha). \tag{15}$$

The approximations given by the NP, using (6) and (7) are

$$VaR_X(\alpha) \simeq \frac{s}{a-1}\left(1 + \sqrt{\frac{a}{a-2}} z_\alpha + \frac{(a+1)}{3(a-3)}\left(z_\alpha^2 - 1\right)\right), \tag{16}$$

$$TVaR_X(\alpha) \simeq \frac{s}{a-1}\left(1 + \frac{\phi(z_\alpha)}{1-\alpha}\left(\sqrt{\frac{a}{a-2}} + \frac{(a+1)}{3(a-3)} z_\alpha\right)\right). \tag{17}$$

The values (exact and approached using NP) of $VaR_X(\alpha)$ and $TVaR_X(\alpha)$ for a $Pa(4,2)$ and confidence levels $0.9, 0.95, 0.99, 0.995$ are included in Table 2.

Table 2. Pareto distribution. Exact and approached values of VaR **and** $TVaR$ **for different confidence levels and** $a = 4$**,** $s = 2$

$E[X] = 2/3 \quad V[X] = 8/9 \quad \gamma_1 = 5\sqrt{2}$				
	$Pa(4,2)$		Normal Power	
α	(14) $VaR_X(\alpha)$	(15) $TVaR_X(\alpha)$	(16) $VaR_X(\alpha)$	(17) $TVaR_X(\alpha)$
0.995	5.521	8.028	9.356	11.670
0.99	4.325	6.433	7.762	10.069
0.95	2.229	3.639	4.112	6.381
0.9	1.557	2.742	2.589	4.820

The differences between the exact values and the approached ones using the NP, $DVaR_X(\alpha)$ and $DTVaR_X(\alpha)$ are,

$$DVaR_X(\alpha) = \frac{s}{a-1}\left(\frac{a-1}{(1-\alpha)^{\frac{1}{a}}} - a - \left(\sqrt{\frac{a}{a-2}} z_\alpha + \frac{(a+1)}{3(a-3)}\left(z_\alpha^2 - 1\right)\right)\right),$$

$$DTVaR_X(\alpha) = \frac{s}{a-1}\left(\frac{a}{(1-\alpha)^{\frac{1}{a}}} - a - \frac{\phi(z_\alpha)}{1-\alpha}\left(\sqrt{\frac{a}{a-2}} + \frac{(a+1)}{3(a-3)} z_\alpha\right)\right).$$

These differences depend on the parameters of the Pareto distribution and the confidence level α; for a fixed a, they are proportional to the expectation of the risk, but they depend in a non-proportional way on the parameter a. The

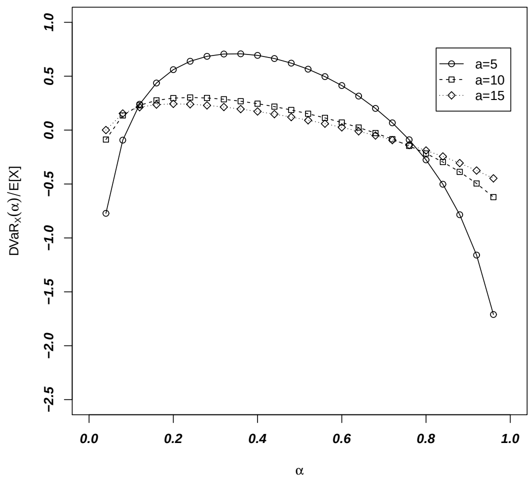

Figure 4. $\frac{DVaR_X(\alpha)}{E[X]}$ as a function of the confidence level α for $a = 5, 10, 15$.

behaviour of $DVaR_X(\alpha)/E[X]$ and $DTVaR_X(\alpha)/E[X]$ with respect to the confidence level α, for different values of a, can be observed in Figures 4 and 5.

The behaviour of these differences are similar for different values of a, but are lower (in absolute value) for higher values of a.

3.3. Lognormal Distribution

Let X be a lognormal distributed r.v., $X \sim Ln(\mu, \sigma^2)$. Then,

$$E[X] = e^{\mu+\sigma^2/2}, V[X] = ae^{2\mu+\sigma^2}, \gamma_1 = (a+3)\sqrt{a}, \text{ being } a = e^{\sigma^2} - 1.$$

The expressions of VaR and $TVaR$ are

$$VaR_X(\alpha) = e^{\mu+\sigma z_\alpha}, \tag{18}$$

$$TVaR_X(\alpha) = \frac{e^{\mu+\sigma^2/2}}{1-\alpha}\left(1 - \Phi(z_\alpha - \sigma)\right). \tag{19}$$

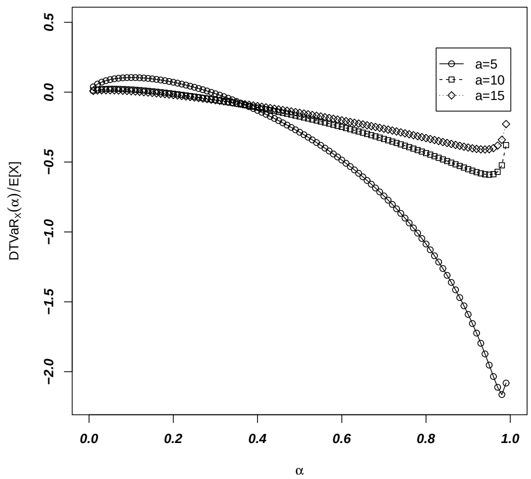

Figure 5. $\frac{DTVaR_X(\alpha)}{E[X]}$ as a function of the confidence level α for $a = 5, 10, 15$.

The approximations given by the NP, using (6) and (7) are

$$VaR_X(\alpha) \simeq e^{\mu+\sigma^2/2}\left(1+\sqrt{a}\left(z_\alpha + \frac{(a+3)\sqrt{a}}{6}\left(z_\alpha^2-1\right)\right)\right), \qquad (20)$$

$$TVaR_X(\alpha) \simeq e^{\mu+\sigma^2/2}\left(1+\sqrt{a}\frac{\phi(z_\alpha)}{1-\alpha}\left(1+\frac{z_\alpha(a+3)\sqrt{a}}{6}\right)\right). \qquad (21)$$

The values (exact and approached using NP) of $VaR_X(\alpha)$ and $TVaR_X(\alpha)$ for a $Ln(1,1)$ and confidence levels $0.9, 0.95, 0.99, 0.995$ are included in Table 3.

The differences between the exact values and the approached ones using the NP, $DVaR_X(\alpha)$ and $DTVaR_X(\alpha)$ are,

$$DVaR_X(\alpha) = e^{\mu+\sigma^2/2}\left(e^{\sigma z_\alpha-\sigma^2/2} - 1 - \sqrt{a}\left(z_\alpha + \frac{(a+3)\sqrt{a}}{6}\left(z_\alpha^2-1\right)\right)\right),$$

$$DTVaR_X(\alpha) = \frac{e^{\mu+\sigma^2/2}}{1-\alpha}\left(\alpha - \Phi(z_\alpha-\sigma) - \sqrt{a}\phi(z_\alpha)\left(1+\frac{z_\alpha(a+3)\sqrt{a}}{6}\right)\right).$$

Table 3. Lognomal distribution. Exact and approached values of *VaR* and *TVaR* for different confidence levels and $\mu = 1$, $\sigma = 1$

$E[X] = 4.48 \quad V[X] = 34.51 \quad \gamma_1 = 6.18$				
	$Ln(1,1)$		Normal Power	
α	(18) $VaR_X(\alpha)$	(19) $TVaR_X(\alpha)$	(20) $VaR_X(\alpha)$	(21) $TVaR_X(\alpha)$
0.995	35.724	51.569	53.738	66.582
0.99	27.836	41.394	44.866	57.686
0.95	14.081	23.261	24.473	37.146
0.9	9.792	17.440	15.901	28.412

For a fixed σ, these differences are proportional with respect to the risk expectation (as in the case of exponential and Pareto distributions) and depends in a non-proportional way on the variance of the Normal distribution associated to the lognormal under study. The behaviour of $DVaR_X(\alpha)/E[X]$ and $DTVaR_X(\alpha)/E[X]$ with respect to the confidence level α, for different values of σ, can be observed in Figures 6 and 7.

The behaviour of these differences are similar for different values of σ, but are lower (in absolute value) for lower values of a.

4. Insurance and Credit Risk Application

In this section, two applications are given: the first concerning the underwriting risk inherent for an insurance company, and the second relative to the credit risk that is present in all financial companies. The two applications begin with the definitions of the risks we are analyzing. The definition of the underwriting risk is extracted from the Directive 2009/138/EC of the European Parliament and of the Council of 25 November 2009 on the taking-up and pursuit of the business of Insurance and Reinsurance (Solvency II) ([10]). We include two definitions of credit risk, the first, for insurance companies comes from [10], and the second, for financial institutions comes from [2] elaborated by the Basel Committee on Banking Supervision.

In each case, we calculate the *VaR* and *TVaR* through simulations and with the NP approximation and we compare them. The credit risk application is used to illustrate the different relation between the different measures of risk.

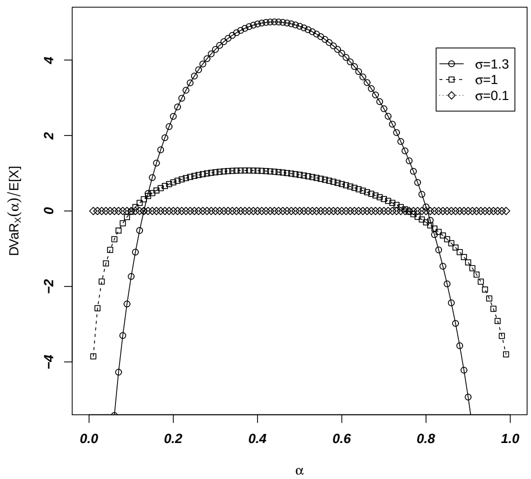

Figure 6. $\frac{DVaR_X(\alpha)}{E[X]}$ as a function of the confidence level α for $\sigma = 1.3, 1, 0.1$.

4.1. Underwriting Risk: Total Cost in a Non-life Insurance Portfolio

The underwriting risk is the main risk of an insurance company/portfolio. It is defined in [10] as *the risk of loss or of adverse change in the value of insurance liabilities, due to inadequate pricing and provisioning assumptions*, so is the characteristic risk in insurance. The risk is related to the randomness of the total claim amount, S, that the insurer must pay in a period due to the insurance contracts.

In this application we are concerned with the underwriting risk of a non-life insurance portfolio. We consider that the total claim amount in a period follows a compound distribution. We assume that the hypotheses of the collective risk theory are fulfilled (see e.g. [4]), that is, we consider that each individual claim amount (which are supposed to be independent and identically distributed) can be modeled by a unique r.v. X, independent of the number of claims in a period

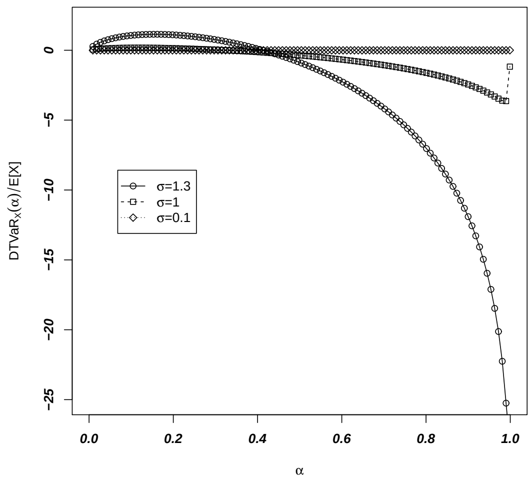

Figure 7. $\frac{DTVaR_X(\alpha)}{E[X]}$ as a function of the confidence level α for $\sigma = 1.3, 1, 0.1$.

in the portfolio, N.

$$S = \begin{cases} 0 & \text{if } N = 0, \\ \sum_{i=1}^{N} X_i & \text{if } N > 0, \end{cases}$$

with $X_i = X$. Compound distributions do not have closed expressions of *VaR* and *TVaR*, then in order to calculate these two risk measures, simulation or approximations are needed. Here we consider the NP approximation that allows us to calculate in an easy way the two values and we perform a numerical analysis comparing the results with the simulated ones in some particular cases (simulations are performed with the package *actuar* ([6]).

If we consider that the number of claims is Poisson distributed, the r.v. total claim amount in a period, S, follows a compound Poisson distribution, $S \sim PoiCom(\lambda, X)$, being λ, the parameter of the Poisson distribution.

The main characteristics of the risk, S, are then ([7])

$$E[S] = \lambda\alpha_1,$$

$$V[S] = \lambda\alpha_2,$$

$$\gamma_1 = \frac{\alpha_3}{\alpha_2^{3/2}\sqrt{\lambda}},$$

being $\alpha_i = E\left[X^i\right]$, $i = 1, 2, 3, \ldots$ the raw moments of the r.v. X (assuming they exist).

The approximations given by the NP, using (6) and (7) are

$$VaR_S(\alpha) \simeq \lambda\alpha_1 + \sqrt{\lambda\alpha_2}\left(z_\alpha + \frac{\alpha_3}{6\alpha_2^{3/2}\sqrt{\lambda}}\left(z_\alpha^2 - 1\right)\right), \tag{22}$$

$$TVaR_S(\alpha) \simeq \lambda\alpha_1 + \sqrt{\lambda\alpha_2}\frac{\phi(z_\alpha)}{1-\alpha}\left(1 + \frac{z_\alpha\alpha_3}{6\alpha_2^{3/2}\sqrt{\lambda}}\right). \tag{23}$$

It is known that the compound Poisson distribution tends to the Normal distribution as λ increases, so we can expect that the values of *VaR* and *TVaR* approached with NP tend towards the simulated results as the mean number of claims increases. This is confirmed by the numbers included in Table 4. The goodness of the NP approximation is reflected in the relative difference of the approached value with respect to the simulated one. These differences in percentage are included in Table 5 (*DV* and *DT* are the relative differences for *VaR* and *TVaR* respectively).

4.2. Credit Risk: An Static Portfolio

Credit risk in insurance companies is defined in [10] as *the risk of loss or of adverse change in the financial situation, resulting from fluctuations in the credit standing of issuers of securities, counterparties and any debtors to which insurance and reinsurance undertakings are exposed, in the form of counterparty default risk, or spread risk, or market risk concentrations*.

Credit risk in financial institutions is defined in [2] as *the potential that a bank borrower or counterparty will fail to meet its obligations in accordance with agreed terms*.

In this application we consider a portfolio first analyzed in [5]. The portfolio is composed by credit insurance contracts of a certain insurance company at 31/12/03. Each exposure is the maximum individual insured value at 31/12/03. This portfolio is described in Table 6.

We consider that if the loss (default) is produced, the loss ratio is 75% (fixed, known and common to all risks in the portfolio), then the loss (if it occurs) is fixed and is the 75% of the exposure for each risk.

Table 4. Compound Poisson distribution. Exponential claim amount $X \sim Exp(1)$. Simulated and approached values of VaR and $TVaR$ for different confidence levels and $\lambda = 1, 50, 100$

$E[S] = 1 \quad V[S] = 2 \quad \gamma_1 = 2.12$				
	$\lambda = 1$ (simulation)		NP approximation	
α	$VaR_X(\alpha)$	$TVaR_X(\alpha)$	$VaR_X(\alpha)$	$TVaR_X(\alpha)$
0.995	7.092	8.412	7.460	8.814
0.99	6.148	7.493	6.496	7.869
0.95	3.917	5.304	4.179	5.614
0.9	2.906	4.329	3.134	4.606
$E[S] = 50 \quad V[S] = 100 \quad \gamma_1 = 0.3$				
	$\lambda = 50$ (simulation)		NP approximation	
α	$VaR_X(\alpha)$	$TVaR_X(\alpha)$	$VaR_X(\alpha)$	$TVaR_X(\alpha)$
0.995	78.501	82.628	78.576	82.644
0.99	75.429	79.723	75.469	79.752
0.95	67.253	72.273	67.301	72.324
0.9	63.099	68.630	63.137	68.674
$E[S] = 100 \quad V[S] = 200 \quad \gamma_1 = 0.21$				
	$\lambda = 100$ (simulation)		NP approximation	
α	$VaR_X(\alpha)$	$TVaR_X(\alpha)$	$VaR_X(\alpha)$	$TVaR_X(\alpha)$
0.995	139.244	144.599	139.245	144.623
0.99	135.086	140.752	135.106	140.792
0.95	124.112	130.850	124.114	130.868
0.9	118.444	125.936	118.445	125.944

In [5], this is considered to be a portfolio of credit insurance contracts, and then they are considered claim amounts, but it can also be considered just as a static portfolio of credits in a financial institution. In this last case we measure its credit risk.

The total risk in the portfolio is modeled with the individual risk model and assuming that the 10 risks are independent. The r.v. that represent the loss for each individual $i = 1, 2, \ldots, 10$, X_i are discrete random variables with values 0

Table 5. Compound Poisson distribution. Exponential claim amount $X \sim Exp(1)$. DV (percentage) and DT for different confidence levels and $\lambda = 1, 50, 100$

	$\lambda = 1$		$\lambda = 50$		$\lambda = 100$	
α	DV	DT	DV	DT	DV	DT
0.995	5.19	4.78	0.09554	0.0193	0.00072	0.01660
0.99	5.66	5.02	0.05303	0.0364	0.01481	0.02842
0.95	6.69	5.84	0.07137	0.0706	0.00161	0.01376
0.9	7.85	6.40	0.06022	0.0641	0.00084	0.00635

Table 6. Characteristics of the portfolio

i	Exposure (in thousand euros) E_i	Rating S&P	Probability of default (percentage) $100 \cdot PD_i$
1	39267	A	0.07
2	38095	A	0.07
3	11717	A-	0.07
4	21305	A+	0.07
5	28001	A+	0.07
6	4531	BB+	1.17
7	6769	BBB-	0.25
8	25962	BBB+	0.25
9	15860	BBB+	0.25
10	7297	CCC	19.96

Source: Carreras (2006), [5].

and $B_i = 0.75 \cdot E_i$ and probabilities $(1 - PD_i)$ and PD_i respectively, that is

$$X_i = \begin{cases} 0, & \text{with probability } (1 - PD_i), \\ B_i, & \text{with probability } PD_i. \end{cases}$$

Let us define S as the total loss in the portfolio

$$S=\sum_{i=1}^{10}E\left[X_i\right].$$

The characteristics of S can be derived from the ones of the individual risks that are in the portfolio, considering the hypotheses of the individual risk model, as follows,

$$E\left[S\right]=\sum_{i=1}^{n}E\left[X_i\right],$$

$$V\left[S\right]=\sum_{i=1}^{n}V\left[X_i\right],$$

$$\gamma_1=\frac{\sum_{i=1}^{n}\mu_3\left(X_i\right)}{\left(\sum_{i=1}^{n}V\left[X_i\right]\right)^{\frac{3}{2}}},$$

being, in this example $n=10$. As the individual risks are dicotomic, $E\left[X_i\right]=B_i\cdot PD_i$, $V\left[X_i\right]=PD_i\cdot(1-PD_i)\cdot B_i^2$, and the third central moment is $\mu_3\left(X_i\right)=B_i\cdot\left(PD_i-3\cdot PD_i^2+2\cdot PD_i^3\right)$.

Then the total loss in the portfolio is described by,

$$E\left[S\right]=1295.88,\ V\left[S\right]=7999935.07,\ \gamma_1=3.66475.$$

In order to obtain estimations of VaR and $TVaR$ of our portfolio, we can simulate the distribution function of the total loss or, we can use the NP approximation. In Table 7, the distribution function of S obtained by simulation is represented. Obviously, S is a discrete r.v.

As S is a discrete r.v., the Value at Risk and the Tail Value at Risk are calculated from its general definitions ((1), (2), (3) and (4)). The results of the different risks measures for a confidence level of 0.95 are

$$VaR_S(0.95)=5472.75,\quad TVaR_S(0.95)=7906.251,$$

$$ES_S(0.95)=121.6750,\quad CTE_S(0.95)=16697.39.$$

As the distribution function of S is discontinuous in the value $VaR_S(0.95)$, i.e. $F_S\left(VaR_S(0.95)\right)=0.98916$, then $TVaR_S(0.95)\neq CTE_S(0.95)$.

Table 7. Distribution function of S by simulation

s	$F_S(s)$	s	$F_S(s)$
0	0.78309	21451.5	0.99787
3398.25	0.79238	22869.75	0.99791
5076.75	0.79429	24399	0.99792
5472.75	0.98916	24548.25	0.99793
8475	0.98918	24944.25	0.99831
8787.75	0.98969	26473.5	0.99847
8871	0.99188	28342.5	0.99848
10549.5	0.99230	28571.25	0.99901
11895	0.99440	29450.25	0.99953
14260.5	0.99447	31366.5	0.99954
15293.25	0.99448	34044	0.99971
15978.75	0.99509	34527	0.99972
16971.75	0.99510	34923	0.99996
17367.75	0.99551	37442.25	0.99997
19471.5	0.99730	40466.25	0.99998
20682.75	0.99731	48921.75	0.99999
21000.75	0.99776	49572	1.00000

If the NP approximation is used, *VaR* and *TVaR* are

$$VaR_S(0.95) = 5948.21,\; TVaR_S(0.95) = 7130.09.$$

In this example, the NP approximation overestimates the *VaR*, but underestimates the *TVaR*. But, we can not generalize these results.

5. Conclusion

In this chapter we have presented a new expression for the Tail Value at Risk when the Normal Power approximation is used. The precision of the approximation is checked with some distributions that have exact expressions for these risk measures: exponential, Pareto and lognormal. Two applications are given: the first concerning the underwriting risk inherent for an insurance company, and the second relative to the credit risk that is present in all financial companies.

The expression obtained for the Tail Value at Risk is a very easy one and depends only on the expectation, the variance and the skewness of the risk. This expression can be applied without knowing the entire distribution (the density or the distribution function) and it is also very easy to compute. Then the important issue is to check whether the approach is good enough.

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Appendix

A.1. Exponential Distribution

Let X be an exponential distributed r.v. with parameter β, $X \sim Exp(\beta)$. The density function is $f_X(x) = \beta e^{-x\beta}$, and the distribution function, $F_X(x) = 1 - e^{-x\beta}$. Then, $E[X] = \frac{1}{\beta}$, $V[X] = \frac{1}{\beta^2}$ and $\gamma_1 = 2$. The expressions of VaR and $TVaR$ are

$$VaR_X(\alpha) = -\frac{\ln(1-\alpha)}{\beta},$$

$$TVaR_X(\alpha) = -\frac{\ln(1-\alpha)}{\beta} + \frac{1}{\beta}.$$

Let us proof the expression of $VaR_X(\alpha)$. From definition of VaR of a continuous random variable

$$VaR_X(\alpha) = F_X^{-1}(\alpha),$$

and from the expression of the distribution function, we have

$$\begin{aligned} \alpha &= F_X(VaR_X(\alpha)), \\ 1-\alpha &= e^{-VaR_X(\alpha)\beta}. \end{aligned}$$

And isolating $VaR_X(\alpha)$, we obtain (10).

Let us now proof the expression of $TVaR_X(\alpha)$,

$$\begin{aligned} TVaR_X(\alpha) &= \frac{1}{1-\alpha}\int_\alpha^1 VaR_X(s)ds = \frac{1}{1-\alpha}\int_{VaR_X(\alpha)}^\infty x dF_X(x) \\ &= \frac{1}{1-\alpha}\int_{-\frac{\ln(1-\alpha)}{\beta}}^\infty x\beta e^{-x\beta}dx = \left\{ \begin{array}{ll} u = x & du = dx \\ dv = \beta e^{-x\beta}dx & v = -e^{-x\beta} \end{array} \right\} \\ &= \frac{1}{1-\alpha}\left(\left[-xe^{-x\beta} \right]_{-\frac{\ln(1-\alpha)}{\beta}}^\infty + \int_{-\frac{\ln(1-\alpha)}{\beta}}^\infty e^{-x\beta}dx \right) \\ &= \frac{1}{1-\alpha}\left(-(1-\alpha)\frac{\ln(1-\alpha)}{\beta} + \frac{(1-\alpha)}{\beta} \right) \\ &= -\frac{\ln(1-\alpha)}{\beta} + \frac{1}{\beta}. \end{aligned}$$

A.2. Pareto Distribution

Let X be a Pareto distributed r.v., $X \sim Pa(a,s)$. The density function is $f_X(x) = \frac{as^a}{(x+s)^{(a+1)}}$, and the distribution function, $F_X(x) = 1 - \left(\frac{s}{s+x}\right)^a$. Then,

$$\begin{aligned} E[X] &= \frac{s}{a-1} \quad (a>1), V[X] = \frac{s^2 a}{(a-1)^2(a-2)} \quad (a>2), \\ \gamma_1 &= \frac{2\sqrt{a-2}(a+1)}{(a-3)\sqrt{a}} \quad (a>3). \end{aligned}$$

The expressions of *VaR* and *TVaR* are

$$VaR_X(\alpha) = \frac{s}{(1-\alpha)^{\frac{1}{a}}} - s,$$

$$TVaR_X(\alpha) = \frac{s}{a-1} + \frac{a}{a-1}VaR_X(\alpha).$$

Let us proof the expression of $VaR_X(\alpha)$. From definition of *VaR* of a continuous random variable

$$VaR_X(\alpha) = F_X^{-1}(\alpha),$$

and from the expression of the distribution function, we have

$$\alpha = F_X(VaR_X(\alpha)),$$
$$1-\alpha = \left(\frac{s}{s+VaR_X(\alpha)}\right)^a,$$
$$(1-\alpha)^{\frac{1}{a}} = \frac{s}{s+VaR_X(\alpha)}.$$

And isolating $VaR_X(\alpha)$, (14) is obtained.

Let us now proof the expression of $TVaR_X(\alpha)$,

$$\begin{aligned}
TVaR_X(\alpha) &= \frac{1}{1-\alpha}\int_\alpha^1 VaR_X(t)dt = \frac{1}{1-\alpha}\int_\alpha^1 \left(\frac{s}{(1-t)^{\frac{1}{a}}} - s\right)dt \\
&= \frac{1}{1-\alpha}\int_\alpha^1 \frac{s}{(1-t)^{\frac{1}{a}}}dt - \frac{1}{1-\alpha}\int_\alpha^1 sdt \\
&= \frac{1}{1-\alpha}\left(\left[\frac{-s(1-t)^{\frac{-1}{a}+1}}{\frac{-1}{a}+1}\right]_\alpha^1 - [st]_\alpha^1\right) \\
&= \frac{a}{a-1}\frac{s}{(1-\alpha)^{\frac{1}{a}}} - s = \frac{a}{a-1}\left(\frac{s}{(1-\alpha)^{\frac{1}{a}}} - s\right) + \frac{sa}{a-1} - s \\
&= \frac{a}{a-1}VaR_X(\alpha) + \frac{s}{a-1}.
\end{aligned}$$

A.3. Lognormal Distribution

The random variable X is lognormal distributed , $X \sim Ln(\mu,\sigma^2)$ if $Y = \ln X$ is $N(\mu,\sigma^2)$. The density function is $f_X(x) = \frac{1}{\sigma x\sqrt{2\pi}}e^{\frac{-(\ln x-\mu)^2}{2\sigma^2}}$, and the distribution function, $F_X(x) = \Phi\left(\frac{\ln x-\mu}{\sigma}\right)$. Then,

$$E[X] = e^{\mu+\sigma^2/2}, V[X] = ae^{2\mu+\sigma^2}, \gamma_1 = (a+3)\sqrt{a}, \text{ being } a = e^{\sigma^2}-1.$$

The expressions of VaR and $TVaR$ are

$$VaR_X(\alpha) = e^{\mu+\sigma z_\alpha},$$

$$TVaR_X(\alpha) = \frac{e^{\mu+\sigma^2/2}}{1-\alpha}(1-\Phi(z_\alpha-\sigma)).$$

Let us proof the expression of $VaR_X(\alpha)$. From definition of VaR of a continuous random variable

$$VaR_X(\alpha) = F_X^{-1}(\alpha),$$

and from the expression of the distribution function, we have

$$\alpha = F_X(VaR_X(\alpha)) = F_{\frac{\ln X-\mu}{\sigma}}\left(\frac{\ln VaR_X(\alpha)-\mu}{\sigma}\right),$$

as $\frac{\ln X-\mu}{\sigma}$ follows a $N(0,1)$, then $\frac{\ln VaR_X(\alpha)-\mu}{\sigma} = z_\alpha$, and isolating $VaR_X(\alpha)$ expression (18) is obtained.

Let us now proof the expression of $TVaR_X(\alpha)$,

$$\begin{aligned}
TVaR_X(\alpha) &= \frac{1}{1-\alpha}\int_{VaR_X(\alpha)}^{\infty} x dF_X(x) \\
&= \frac{1}{1-\alpha}\int_{VaR_X(\alpha)}^{\infty} \frac{x}{\sigma x\sqrt{2\pi}} e^{\frac{-(\ln x-\mu)^2}{2\sigma^2}} dx \\
&= \left\{ z = \frac{\ln x-\mu}{\sigma} \right\} = \frac{1}{1-\alpha}\int_{z_\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\sigma z+\mu} e^{\frac{-z^2}{2}} dz \\
&= \frac{1}{1-\alpha} e^{\mu+\sigma^2/2}\int_{z_\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-(z-\sigma)^2}{2}} dz \\
&= \frac{1}{1-\alpha} e^{\mu+\sigma^2/2}\left(1-\Phi(z_\alpha-\sigma)\right).
\end{aligned}$$

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