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Chapter 15

SINGULARITIES OF THE NAVIER-STOKES EQUATIONS IN DIFFERENTIAL FORM AT THE INTERFACE BETWEEN AIR AND WATER

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Abstract

The properties of the Navier-Stokes equations are extremely important for theoretical and numerical studies of fluid flow. This chapter investigates the properties of the differential and integral forms of the Navier-Stoke equations for immiscible air-water flow. The analysis reveals that unlike other fluids the immiscible air-water flow is so special that the Navier-Stokes equations in the differential form have singularities at the interface due to the continuous movement of interface and discontinuous density. In contrast to the differential form, the Navier-Stokes equations in integral form hold well every where including at the interface, indicating the integral form can be widely used in the computation of air-water flow.

Keywords: Navier-Stokes equations, air-sea interaction, wind-wave interaction, air-water flow

AMS Subject Classification: 53D, 37C, 65P

1. Introduction

Immiscible air-water fluids exist widely and have many applications in practical problems, for example, the air-water flow in river and ocean. In an immiscible interfacial fluid the fluid domain is divided into sub-domains by interfaces, and each sub-domain is occupied by a single phase fluid with constant density and viscosity. The most distinct feature of an immiscible air-water fluid is that the thickness of the interface is zero and the fluid

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density and viscosity suddenly change from one constant to another across the interface. For example, across the interface between air and water the density suddenly jumps from around $\rho = 1.0kg/m^3$ in the air to $\rho = 1000kg/m^3$ in the water. Other features of viscous immiscible air-water flow are that the interface moves continuously and the velocity, pressure, and shear stress are continuous functions across the interface.

The Navier-Stokes equations in differential form together with finite difference schemes have been widely used in order to investigate air-water flow in last three decades and two-phase model have been developed. In the two-phase model the domains occupied by the water and air is treated as a single domain and the Navier-Stokes equations in differential form are solved in the air and water simultaneously. The two-phase model has been widely used: a comprehensive review of this approach was given by Scardovelli and Zaleski (1999). Since then a large volume of papers have been published based on the two-phase model; the publications of Tryggvason et al. (2001) and Lafrait et al. (2014) are examples.

Despite intensive research in the past, seeking analytical or numerical solutions of air-water flow or interfacial (free surface) flow is still of great importance in theoretical research and practical applications. The Navier-Stokes equations are well known in fluid dynamics, but one has not paid their attention to some properties of air-water flow. It will be seen in this paper that the equations with singular derivatives are not suitable for finite difference schemes to be applied for the numerical simulation of fluid flow. The fact that the Navier-Stokes equations in differential form and finite difference schemes are widely used in the two-phase model indicates that the singularities of the Navier-Stokes equations in differential form should be discussed, the purpose of this paper is to reveal these properties and the paper is organised as follows: § 2 reveals the singularities at the interface of the Navier-Stokes equations in differential form in the two-phase model, § 3 analyses the validation of the the Navier-Stokes equations in integral form, and § 4 draws the conclusions.

2. The Navier-Stokes Equations in Differential Form and Singularities at Interface

A two-dimensional air-water flow is shown in Figure 1. The solution domain is occupied by two immiscible incompressible, viscous fluids 1 and 2 separated by an interface. $\mathbf{v} = u\mathbf{i} + v\mathbf{j}$ is the velocity vector and p is the pressure. In the two-phase model the density and viscosity of the fluid are denoted by ρ and μ. In fluid 1 $\rho = \rho_1$ and $\mu = \mu_1$, while in fluid 2 $\rho = \rho_2$ and $\mu = \mu_2$. In the differential form, the incompressibility condition is expressed by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

the mass conservation is written as

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0, \tag{2}$$

and the momentum equations for (u, v) are given by

$$\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho uu)}{\partial x} + \frac{\partial (\rho vu)}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(\mu \frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(\mu \frac{\partial u}{\partial y}) \tag{3}$$

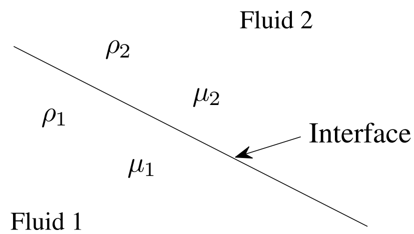

Figure 1. The domain occupied by two immiscible fluids separated by an interface.

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho vv)}{\partial y} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}(\mu\frac{\partial v}{\partial x}) + \frac{\partial}{\partial y}(\mu\frac{\partial v}{\partial y}) - \rho g \tag{4}$$

where g is gravitational acceleration.

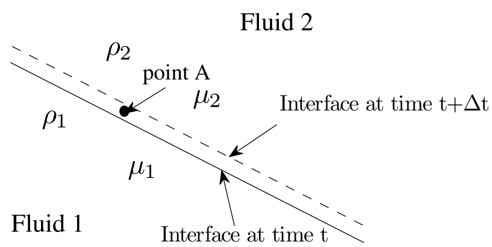

Figure 2. Point A is in fluid 1 at time t and in fluid 2 at time $t + \Delta t$.

First, we analyse the singularities of the time-derivatives at the interface. Assume point A is a fixed point in the space shown in Figure 1, the interface is continuously moving. At time t the interface is at the location indicated by the solid line, point A is inside fluid 2, therefore $\rho(t) = \rho_2$, $u(t) = u_2$ and $v(t) = v_2$, respectively. During time period Δt the interface moves to the location indicated by the dashed line. Point A is inside fluid 1 at time $t + \Delta t$, when the density and velocities at point A are $\rho(t+\Delta t) = \rho_1$, $u(t+\Delta t) = u_1$ and $v(t + \Delta t) = v_1$. We assume point A always stays between the two interfaces, hence when Δt tends to zero the two interfaces will move to point A, therefore

$$\frac{\partial \rho}{\partial t} = \lim_{\Delta t \to 0} \frac{\rho(t+\Delta t) - \rho(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\rho_1 - \rho_2}{\Delta t} = \infty \tag{5}$$

Equation (5) indicates that the time derivative of density $\frac{\partial \rho}{\partial t}$ is infinite at the interface because when an interface is passing an observer at point A, density ρ at the observer suddenly changes from $\rho = \rho_2$ to $\rho = \rho_1$ within an infinitesimal period of time.

Now we calculate the time derivative $\frac{\partial(\rho u)}{\partial t}$ in equation (3) when the interface is passing the fixed point A. Since velocities u and v are continuous functions they can be assumed to be constant during Δt, namely $u_1 = u_2 = u$ and $v_1 = v_2 = v$. Thus we have

$$\frac{\partial(\rho u)}{\partial t} = \lim_{\Delta t \to 0} \frac{\rho(t+\Delta t)u(t+\Delta t) - \rho_(t)u(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\rho_1 u_1 - \rho_2 u_2}{\Delta t} = \lim_{\Delta t \to 0} u \frac{\rho_1 - \rho_2}{\Delta t} = \infty \tag{6}$$

Similarly we have

$$\frac{\partial(\rho v)}{\partial t} = \lim_{\Delta t \to 0} \frac{\rho(t+\Delta t)v(t+\Delta t) - \rho(t)v(t)}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{\rho_1 v_1 - \rho_2 v_2}{\Delta t} = \lim_{\Delta t \to 0} v \frac{\rho_1 - \rho_2}{\Delta t} = \infty \tag{7}$$

Therefore, we conclude that all the time-derivatives in the Navier-Stokes equations (2)-(4) are infinite at the interface, namely the time-derivatives in the Navier-Stokes equations in differential form are singular.

Second, we analyse the singularities in the advection terms on the left hand side of equations (2)-(4). As shown in Figure 1 velocities u and v are continuous functions in the solution domain including at the interface, but the density ρ has a jump across the interface. Thus ρu, ρv in equation (2), ρuu, ρvu and ρvv in equations (3) and (4) are all discontinuous functions at the interface. It is well known that the derivative at a discontinuity does not exist, hence all the $x-$ and $y-$ derivatives $\frac{\partial(\rho u)}{\partial x}$, $\frac{\partial(\rho v)}{\partial y}$, $\frac{\partial(\rho uu)}{\partial x}$, $\frac{(\rho uv)}{\partial y}$, $\frac{\partial(\rho vu)}{\partial x}$ and $\frac{\partial(\rho vv)}{\partial y}$ in equations (2)-(4) do not exist at the interface. Therefore, we have revealed that all terms on the left hand side of (2)-(4) are singular at the interface and we can see that it is the discontinuous density that causes the singularities in the time-derivatives and space-derivatives.

The singularities at the interface may cause problems in numerical simulation. For example, if the time-derivative in equation (6) is approximated by a finite difference scheme then

$$\frac{\partial(\rho u)}{\partial t} \approx \frac{\rho(t+\Delta t)u(t+\Delta t) - \rho(t)u(t)}{\Delta t} = \frac{\rho_1 u_1 - \rho_2 u_2}{\Delta t} \tag{8}$$

From equation (8) we can see that if ρ_1 and ρ_2 are two constants and velocity is a continuous function, then the value of the finite difference increases as the time-step Δt decreases and the numerical derivative becomes increasingly large, consequently, the numerical solution may become increasing inaccurate. For the advection terms, for example $\frac{\partial(\rho u)}{\partial x}$, since the space-derivative does not exist at the interface, it is impossible for any finite difference scheme to approximate the non-existing derivative. In fact, the spurious velocity and current near the interface appeared in the publications of Scardovelli and Zaleski (1999), Tryggvason et al. (2001) and Lafrait et al. (2014) are caused by the singularities. When using the differential form of the equations One may suggest using a moving frame. If the shape of the interface does not change in the moving frame then the time derivatives become zero, and the time-derivatives in equations (2)-(4) are valid at the interface. However, the discontinuity of the density still causes the non-existence of the space-derivatives in the moving frame. In general, the shape of the interface constantly changes even in a moving frame. For example, a breaking wave constantly changes its shape during a breaking process and the time-derivatives are always infinite at the interface even in a moving frame.

3. The Navier-Stokes Equations in Integral Form

Equivalent to the differential form of the Navier-Stokes equations (1)-(4) are the Navier-Stokes equations in integral form (Hirsch 1998, Ferziger 2002 and Wen 2013). For a finite volume Ω in the solution domain the air and water are assumed incompressible, and then the volume of the fluid within the finite volume does not change. Therefore, the fluid volume entering the finite volume through the surface of the finite volume equals that leaving the finite volume through the surface of the finite volume. Corresponding to equation (1) the continuity equation for the volume conservation is given by

$$\int_S (\mathbf{n} \cdot \mathbf{v}) dS = \mathbf{0} \tag{9}$$

where S is the surface of the finite volume, Ω is the volume of the finite volume, and $\mathbf{n} = n_x \mathbf{i} + n_y \mathbf{j}$ is the unit vector outward from the surface of the finite volume.

When a finite volume Ω contains an interface which is continuously moving, the mass of fluid within the finite volume varies continuously with time t. The mass conservation stated that the rate of change of the total fluid mass within the finite volume is caused by the flow rate of fluid through the surface of the finite volume. Corresponding to equation (2) the continuity equation for mass conservation is then expressed as

$$\frac{\partial}{\partial t} \int_\Omega \rho d\Omega + \int_S (\rho \mathbf{n} \cdot \mathbf{v}) dS = \mathbf{0}. \tag{10}$$

Similarly the momentum equation for u is given by

$$\frac{\partial}{\partial t} \int_\Omega (\rho u) d\Omega + \int_S (\rho \mathbf{n} \cdot \mathbf{v}) u dS = -\int_\mathbf{S} n_x p dS + \int_\mathbf{S} \tau_\mathbf{u} dS \tag{11}$$

and the momentum equation for v is given by

$$\frac{\partial}{\partial t} \int_\Omega (\rho v) d\Omega + \int_S (\rho \mathbf{n} \cdot \mathbf{v}) v dS = -\int_\mathbf{S} n_y p dS + \int_\mathbf{S} \tau_\mathbf{v} dS - mg, \tag{12}$$

where $m = \int_\Omega \rho d\Omega$ is the total fluid mass within the finite volume, $\tau_u = \mu \frac{\partial u}{\partial n}$, $\tau_v = \mu \frac{\partial v}{\partial n}$ are the shear stress at the surface of the finite volume and $\frac{\partial u}{\partial n}$ and $\frac{\partial v}{\partial n}$ are directional derivatives along $\mathbf{n}$ direction.

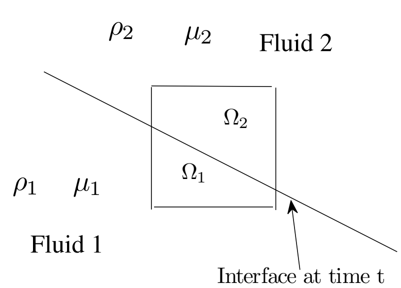

Figure 3. Interface is continuously moving within a finite volume.

We are going to find out whether singularities occur in equations (10)-(12). We consider a finite volume Ω in Figure 3. A continuously moving interface separates Ω into two

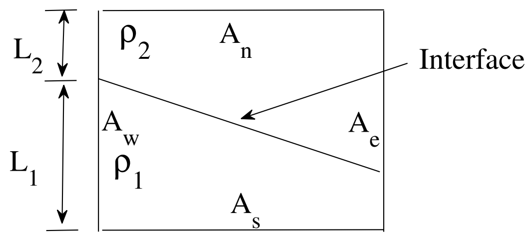

Figure 4. Rectangular finite volume.

volumes Ω_1 and Ω_2, thus $\Omega = \Omega_1 + \Omega_2$. The total fluid mass m within the finite volume is calculated by

$$m = \int_\Omega \rho d\Omega = \int_{\Omega_1+\Omega_2} \rho d\Omega = \int_{\Omega_1} \rho d\Omega + \int_{\Omega_2} \rho d\Omega = \rho_1\Omega_1 + \rho_2\Omega_2 \tag{13}$$

Clearly volumes Ω_1 and Ω_2 are continuous functions of time t due to the continuous movement of the interface, hence m is a continuous function.

We have

$$\frac{\partial}{\partial t}(\int_\Omega \rho d\Omega) = \frac{\partial m}{\partial t} = \rho_1\frac{\partial \Omega_1}{\partial t} + \rho_2\frac{\partial \Omega_2}{\partial t} \tag{14}$$

From the mean value theorem of integration we have

$$\frac{\partial}{\partial t}\int_\Omega (\rho u) d\Omega = \frac{\partial}{\partial t}(u_p\int_\Omega \rho d\Omega) = \frac{\partial}{\partial t}(u_p m) = \rho_1\frac{\partial}{\partial t}(u_p\Omega_1) + \rho_2\frac{\partial}{\partial t}(u_p\Omega_2) \tag{15}$$

where u_p is the mean velocity in the finite volume.

Similarly we have

$$\int_\Omega \frac{\partial(\rho v)}{\partial t} d\Omega = \frac{\partial}{\partial t}(v_p m) = \rho_1\frac{\partial}{\partial t}(v_p\Omega_1) + \rho_2\frac{\partial}{\partial t}(v_p\Omega_2) \tag{16}$$

Since Ω_1, Ω_2 and the velocities of the viscous fluid are continuous functions, then $u_p\Omega_1$, $u_p\Omega_2$, $v_p\Omega_1$ and $v_p\Omega_2$ are also continuous functions, therefore, the time derivatives $\frac{\partial\Omega_1}{\partial t}$, $\frac{\partial\Omega_2}{\partial t}$, $\frac{\partial(u_p\Omega_1)}{\partial t}$, $\frac{\partial(u_p\Omega_2)}{\partial t}$, $\frac{\partial(v_p\Omega_1)}{\partial t}$, $\frac{\partial(v_p\Omega_2)}{\partial t}$ are all valid.

We now prove that space integrations in equations (10)-(12) can be easily calculated.

For a rectangular finite volume shown in Figure 4, from equation (10) we have

$$\int_S (\rho\mathbf{n}\cdot\mathbf{v}) dS = \int_{\mathbf{A_e}} \rho u dS - \int_{\mathbf{A_w}} \rho u dS + \int_{\mathbf{A_n}} \rho v dS - \int_{\mathbf{A_s}} \rho v dS = F_e - F_w + F_n - F_s \tag{17}$$

where F_e, F_w, F_n and F_s are the mass fluxes passing through surfaces A_e, A_w, A_n and A_s and they are given by

$$\left.\begin{array}{ll} F_e = \int_{A_e} \rho u dS & F_w = \int_{A_w} \rho u dS \\ F_n = \int_{A_n} \rho v dS & F_s = \int_{A_s} \rho v dS \end{array}\right\} \tag{18}$$

Now we calculate the mass fluxes or $F's$. We use face A_w in Figure 4 as an example. Suppose L_1 is exposed to fluid 1 and L_2 is exposed to fluid 2, then from the mean value theorem of integration we have

$$\begin{aligned} F_w &= \int_{A_w} \rho u dS = u_w \int_{A_w} \rho dS = u_w \int_{L_1+L_2} \rho dS \\ &= u_w(\int_{L_1} \rho dS + \int_{L_2} \rho dS) = u_w(\rho_1 L_1 + \rho_2 L_2) \end{aligned} \tag{19}$$

Then equations in (18) are written as:

$$\left.\begin{aligned} F_e = u_e(\rho_1 L_1 + \rho_2 L_2)_e \qquad F_w = u_w(\rho_1 L_1 + \rho_2 L_2)_w \\ F_n = v_n(\rho_1 L_1 + \rho_2 L_2)_n \qquad F_s = v_s(\rho_1 L_1 + \rho_2 L_2)_s \end{aligned}\right\} \tag{20}$$

where u_e, u_w, v_n and v_s are the mean velocities at surfaces A_e, A_w, A_n and A_s. Wen (2012) made a further discussion on why the mass flues given by (20) can avoid the spurious velocity near the interface.

It is clear that L_1 and L_2 are continuous functions of time t due to the continuous movement of the interface, and then the mass fluxes $F's$ are continuous functions. Substituting equations (16) and (17) into equation (10) leads to

$$\rho_1 \frac{\partial \Omega_1}{\partial t} + \rho_2 \frac{\partial \Omega_2}{\partial t} = -(F_e - F_w + F_n - F_s) \tag{21}$$

Equation (21) proves that $\frac{\partial \Omega_1}{\partial t}$ and $\frac{\partial \Omega_2}{\partial t}$ are continuous functions since the mass fluxes $F's$ are continuous functions.

From the mean value theorem of integration, the term $\int_S (\rho \mathbf{n} \cdot \mathbf{v}) u dS$ in equation (11) is given by

$$\begin{aligned} \int_S (\rho \mathbf{n} \cdot \mathbf{v}) u dS &= \int_{A_e} \rho u u dS - \int_{A_w} \rho u u dS + \int_{A_n} \rho v u dS - \int_{A_s} \rho v u dS \\ &= u_e \int_{A_e} \rho u dS - u_w \int_{A_w} \rho u dS + u_n \int_{A_n} \rho v dS - u_s \int_{A_s} \rho v dS \\ &= u_e F_e - u_w F_w + u_n F_n - u_s F_s \end{aligned} \tag{22}$$

therefore, the momentum flux terms given by equation (22) are continuous functions since $F's$ and u are continuous functions. Similarly, we can work out the momentum flux terms in (12).

The pressure term in equation (11) is given by

$$-\int_S n_x p dS = -\int_{A_e} p dS + \int_{A_w} p dS \tag{23}$$

For viscous fluid the pressure in Equation (23) is a continuous function everywhere in the solution domain. Equation (23) indicates that the total pressure on the interfaces of the finite volume can be calculated by the integration of the pressure.

The stress term in equation (11) is given by

$$\int_S \tau_u dS = \int_{A_e} \tau_u dS + \int_{A_w} \tau_u dS + \int_{A_n} \tau_u dS + \int_{A_s} \tau_u dS \tag{24}$$

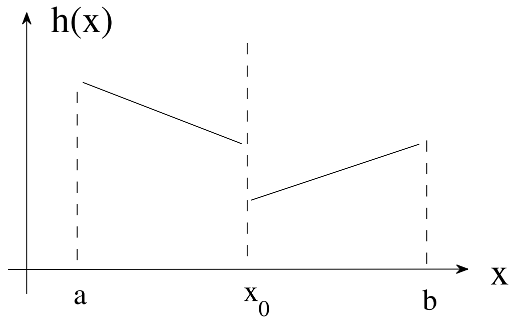

Figure 5. Piecewise continuous function h(x).

The shear stresses in a viscous fluid are continuous functions even at the interface. Similarly, we can work out the shear stress terms in (12).

From equation (20) we can see the density is represented by the constants ρ_1 and ρ_2, it is no longer a variable in the integral form. The new variables are Ω_1, Ω_2, L_1 and L_2 which are all continuous functions due to the continuous movement of the interface. Therefore, all of (10)-(12) are valid.

In order to further explain the difference between the differential and integral forms we analyse the difference between integration and differentiation. Shown in Figure 5 is a piecewise continuous function $h(x)$ with a jump at point x_0. Clearly the derivative $\frac{dh}{dx}$ does not exist at x_0, namely $h(x)$ is not differentiable at point x_0, whereas the integration $\int_a^b h(x)dx$ is always integrable even if function $h(x)$ is discontinuous within interval [a,b]. From this example we can see that the Navier-Stokes equations in the integral form are always valid because the integration well adapts to discontinuity whereas the Navier-Stokes equations in the differential form are not valid because differentiation does not tolerate discontinuity.

Conclusion

A detailed investigation of the properties of the Navier-Stokes equations for an immiscible viscous multiphase fluid is performed. In the Navier-Stokes equations in differential form, the continuous movement of the interface and discontinuous density cause infinite time derivatives and and non-existing space derivatives at the interface. These singularities at the interface are important features of the Navier-Stokes equations in the differential form. All these shortcomings at the interface indicate the the immiscible multiphase fluid is a very special fluid. The main reason for the singularities to occur at the interface is that the differentiation does not tolerate the discontinuity. It is clear that the equations with singularities are difficult to use in seeking the numerical solution at the interface. Therefore, when the Navier-Stokes equations in the differential form in the two-phase model are deployed the accuracy of the numerical solution is questionable since no finite difference scheme is able to approximate the derivatives which tend to infinity or do not exist.

The analysis in this chapter proves that the terms in the Navier-Stokes equations in integral form hold very well and are continuous in solution domain including at the interface. Therefore, the Navier-Stokes equations in integral form are recommended when deploying the two-phase model.

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